Forcing and the Continuum Hypothesis

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1 Introduction

1.1 Independence

Example. In Euclidean geometry, Euclid lays out five postulates, and in particular he lays out the fifth postulate:

(v) Parallel postulate: For any given line, and any given point not on that line, there is exactly one line through the given point which does not meet the given line.

People kind of hated this axiom, it was significantly more complicated than the other axioms, so people wanted to prove it using the other axioms. However, in the 19th Century Gauss, Lobachevsky, and others proved that one can consistently have axioms (i)-(iv), and not (v).

So (v) is independent of (i)-(iv).

Example. Let $\varphi := \exists x\,(x^2 = 2)$. We have that $\mathbb{Q}$ is a field satisfying $\neg\varphi$, but that $\mathbb{Q}[\sqrt{2}]$ is a field satisfying φ. So φ is independent of the field axioms.

Example. The number of roots of $x^4 = 2$, in $\mathbb{Q}$ the are no roots to this. In $\mathbb{R}$ there are two roots to this, and in $\mathbb{C}$ there are 4 roots to this.

However, set theory was set up to encapsulate *all* mathematics. So perhaps set theory is immune to this type of problem, and everything can either be proved or disproved.

Gödel's incompleteness theorems showed that set theory is *not* immune to this type of problem, and the consistency of ZFC is independent of ZFC (assuming that ZFC is in fact consistent).

1.2 Continuum problem

Now cast your mind back to Cantor, and recall

Theorem (Cantor)**.**

$$|\mathbb{N}| < \mathcal{P}(\mathbb{N}).$$

Then we have a natural question, is there a set X such that $|\mathbb{N}| < |X| < |\mathcal{P}(\mathbb{N})|$? The **Continuum hypothesis** is the claim that if $X \subseteq \mathcal{P}(\mathbb{N})$ is infinite, then $|X| = |\mathbb{N}|$ or $|X| = |\mathcal{P}(\mathbb{N})|$. That is to say, $2^{\aleph_0} = \aleph_1$. This question inspired a lot of maths, with some key results being:

Theorem (Cantor, 1883)**.** *Any closed subset of* $\mathbb{R}$ *satisfies* CH.

Theorem (Alexandrov/Hausdorff, 1916)**.** *Any Borel set of* $\mathbb{R}$ *satisfies* CH*.*

Theorem (Suslin, 1930)**.** *Any anaylytic subset of* $\mathbb{R}$ *satisfies* CH*.*

Theorem (Gödel, 1938)**.** $\mathrm{Con}(\mathsf{ZF}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + \mathsf{CH})$.

Theorem (Cohen, 1963)**.** $\mathrm{Con}(\mathsf{ZF}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

1.3 Systems of Set Theory

The language of set theory, $\mathcal{L} := \mathcal{L}_{\in}$, consists of:

- First-order predicate logic
- Two binary symbols: $\in$ and $=$.
- Set variables: $\nu_1, \ldots, \nu_n, \ldots$
- Logical connectives: $\vee$, $\neg$.
- Brackets: (, and).
- Existential quantifier: $\exists$

Remark. $\wedge$, $\rightarrow$, and $\forall$ can be defined in terms of $\vee$, $\neg$, and $\exists$.

Definition 1.1 (Free variable/Bound variable)**.** A variable is **bound** in φ if it is the occurrence of an x which is in a quantifier, or if it is within the scope of such a quantifier.

Otherwise, we say a variable is **free**, or occurs freely. We write $\mathrm{Fr}(\varphi)$ to denote the set of free variable in φ.

Remark.
- We write $\varphi(u_0, \ldots, u_n)$ to emphasise the dependence of φ on $u_1, \ldots, u_n$.
- We allow ourselves to freely chage variables, i.e. $\varphi(v_0, \ldots, v_n)$ denotes the same formula with different variables.
- We assume the substituted variables are free.
- However, writing $\varphi(u_0, \ldots, u_n)$ will not imply that u occurs freely, or even occurs at all.

1.4 Theories

The theories we consider are

- ZF consisting of the axioms: Extension, Pairing, Unions, Empty set, Foundation, Separation, Replacement, Power set, and Infinity.
- ZFC is ZF with the Axiom of Choice.
- ZF^- is ZF without the power set axiom. (When we say ZF without the power set axiom, we have to change replacement to the axiom scheme of collection, since otherwise we can prove the power set axiom).
- ZFC^- is ZF^- + Well-ordering principle.
- Z is ZF $\setminus$ Replacement.

Remark. Our main background theory will be ZF.

2 Classes

Definition (Definable class/Proper Class)**.** A class X is **definable** over $\mathcal{M}$ if there exists a formula φ, and sets $a_1, \ldots, a_n \in M$ such that

$$\forall z \in \mathcal{M}\ (z \in X \Leftrightarrow \varphi(z, a_1, \ldots, a_n)).$$

A class X is **proper** (over $\mathcal{M}$) if $X \notin \mathcal{M}$.

We will assume that all classes are definable.

Example.

- - $V = \{x : x = x\}$
  - $R = \{x : x \notin x\}$
  - Ord

  are all definable proper classes
- Any set is a class.
- Classes are heavily dependent on the model. If $\mathcal{M} = \mathsf{Z}$ then Ord $= \mathcal{M} = \mathsf{Z}$.

2.1 Adding defined functions

The objects 0, $\perp$, $\subseteq$, $\cap$ all do not exist in our language.

Definition (defined n-ary predicate/defined n-ary function symbol)**.** Assume that $\mathcal{L} \subseteq \mathcal{L}'$, and T is a set of sentences of L. Then

- p is a **defined n-ary predicate** symbol over T if there is a formula φ in $\mathcal{L}$ such that

$$T \vdash \forall x_1, \ldots, x_n\, (p(x_1, \ldots, x_n \leftrightarrow \varphi(x_1, \ldots, x_n))).$$

- f is a **defined n-ary function symbol** over T if there exists a formula φ in $\mathcal{L}$ such that for any $x_1, \ldots, x_n$,

$$p(x_1, \ldots, x_n) = y$$

iff $T \vdash \varphi(x_1, \ldots, x_n, y)$ or $T \vdash \forall x_1, \ldots, x_n \exists! y, \;\; \varphi(x_1, \ldots, x_n, y)$.

Definition (Extension by definitions)**.** A set of sentences T' of $calL'$ is an extension by definitions of T over $\mathcal{L}$ iff $T' = T \cup S$ when

$$S = \{\varphi_s \mid s \in \mathcal{L}' \setminus \mathcal{L}\}$$

and each φ_s is a definition of s in the language $\mathcal{L}$ over T.

Proposition 2.1. *The following are defined over* ZF*:* 0, 1, $\subseteq$, $\cap$, $\mathcal{P}$, $\cup$.

Theorem. *Suppose $\mathcal{L} \subseteq \mathcal{L}'$, and T is a set of sentences of $\mathcal{L}$, and T' is an extension by definitions of T over $\mathcal{L}$. Then we have*

(1) Conservativity: If φ is a sentence of $\mathcal{L}$. Then $T \vdash \varphi$ iff $T' \vdash \varphi$.

(2) Abbreviations: If φ is a formula of $\mathcal{L}'$, then there exists a formula $\hat{\varphi}$ of $\mathcal{L}$, with

$$\mathrm{Fr}(\varphi) = \mathrm{Fr}(\hat{\varphi}),$$

and $T \vdash \forall x\, (\varphi \leftrightarrow \hat{\varphi})$.

Example. The intersection of two sets, $a \cap b$, can be defined as *the* set c such that

$$\forall x\, (x \in c \leftrightarrow x \in a \cap x \in b)$$

This only makes sense if there is a *unique* c satisfying this.

For example, let

$$\mathcal{M} = \{a, c, d, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}.$$

Both $\{a\}$ and $\{a, b\}$ satisfy $\varphi(c)$.

Lecture 2

Assume that there is a countable transitive model of ZFC, $\mathcal{M}$. Then $|\mathbb{R} \cap M|$ is countable, so $\exists v \in \mathbb{R} \setminus M$. So v is a *proper* class over $\mathcal{M}$, but v is not *definable* over $\mathcal{M}$ (if it were, then replacement would but it in $\mathcal{M}$).

Remark.

- If it's not definable, then we "can't talk about it" in $\mathcal{L}_{\in}$.

- So the only proper classes that affect our theory are the definables ones, Ord, V, etc.
- We can use formulae of the form:

$$\exists C\,(C \text{ is a class} \wedge \forall x(x \in C \to \dots)).$$

  This is an abbreviation for "There is a formula θ such that $\forall x$, $(\theta(x) \to \dots)$".

Suppose $\mathcal{M}$ is a model of ZF. Let $\mathcal{D}$ be the collection of definable classes over $\mathcal{M}$. Then $\mathcal{D}$ is a set in V, and $(\mathcal{M}, \mathcal{D})$ is a model of a second-order version of ZF, which is known as Gödel-Bernays Set theory.

3 Absoluteness

Observe that definitions often appear to give the same set, regardless of which model of ZFC we are working in.

Examples. 1. $\{x : x \neq x\}$ gives the empty set.

2. $\{x : x = a \vee x = b\}$ gives the pair set.

However, other definitons do not give the same set, for example $\mathcal{P}(\mathbb{N})$ is not the same in the countable transitive model as it is in a more "normal" model.

Therefe, to discuss which definitions are definite, we need to define what it means for φ to hold in a structure.

3.1 Relativisation

Definition 3.1 (Bounded Quantifier)**.** $\forall x \in a(\dots)$ is an abbreviation for $\forall x(x \in a \to \dots)$.

$\exists x \in a$, $\varphi(x)$ is an abbreviation for $\exists x\,(x \in a \wedge \varphi(x))$.

Definition 3.2 (Relativisation)**.** Let W be a class, then we define φ^W by recursion as

- $(x \in y)^W \equiv (x \in y)$
- $(x = y)^W \equiv (x = y)$
- $(\varphi \vee \psi)^W \equiv \varphi^W \vee \psi^W$
- $(\neg\varphi)^W \equiv \neg(\varphi)^W$
- $(\exists x\, \varphi)^W \equiv \exists x \in W\, \varphi^W$.

Proposition 3.3.

$$(\varphi \wedge \psi)^W \equiv \varphi^W \wedge \psi^W$$
$$(\varphi \to \psi)^W \equiv \varphi^W \to \psi^W$$
$$(\forall x, \varphi(x))^W \equiv \forall x \in W \varphi^W(x).$$

Proof. Exercise. □

Proposition 3.4. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}$ is a definable class over $\mathcal{N}$, then the relation $\mathcal{M} \models \varphi$ if first-order expressible in $\mathcal{N}$*

Proof. Suppose $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is definable by θ (so $\forall z \in \mathcal{N}$, $\theta(z) \Leftrightarrow z \in \mathcal{M}$).

Then we claim that $(\mathcal{N}, \in) \models \varphi^{\mathcal{M}}$ iff $(\mathcal{M}, \in) \models \varphi$. $\mathcal{N} \models (x \in y)^{\mathcal{M}}$ iff $\mathcal{N} \models x \in y$ (and $x, y \in \mathcal{M}$) iff $\theta(x)$, $\theta(y)$, $\langle x, y\rangle \in (\in \cap \mathcal{M}^2)$ i.e. $\mathcal{M} \models x \in y$.

Suppose the claim holds for φ and ψ, then

$$\begin{aligned} N \models (\varphi \vee \psi)^M &\text{ iff } N \models \varphi^M \vee \psi^M \\ &\text{ iff } N \models \varphi^M \text{ or } N \models \psi^M \\ &\text{ iff } \mathcal{M} \models \varphi^M \text{ or } M \models \psi^{\mathcal{M}} \end{aligned}$$

Also have that $\mathcal{N} \models (\exists x \varphi(x))^{\mathcal{M}}$ iff $\mathcal{N} \models \exists x\, (x \in \mathcal{M} \wedge \varphi^M(x))$ iff there is some $x \in \mathcal{N}$ such that $\mathcal{N} \models x \in \mathcal{M}$ and $\mathcal{N} \models \varphi^M(x)$ iff there is some $x \in \mathcal{N}$ such that $\theta(x) \wedge \mathcal{M} \models \varphi(x)$. So $\mathcal{M} \models \exists x \varphi(x)$. □

Definition 3.5 (Upwards/Downwards absolute, absolute)**.** Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are classes, and $\varphi(u_1, \ldots, u_n)$ isa formula, then

- φ is upwards absolute for $\mathcal{M}$, $\mathcal{N}$ iff

$$\forall x_1, \ldots, \forall x_n \in M(\varphi^M(x_1, \ldots, x_n) \to \varphi^{\mathcal{N}}(x_1, \ldots, x_n))$$

- φ is downwards absolute for $\mathcal{M}$, $\mathcal{N}$ iff

$$\forall x_1, \ldots, x_n \in M\, (\varphi^{\mathcal{N}}(x_1, \ldots, x_n) \to \varphi^{\mathcal{M}}(x_1, \ldots, x_n))$$

- φ is absolute if it is both upwards and downwards absolute.

Examples.

- If $\mathcal{N} = V$, ust say φ is absolute for $\mathcal{M}$

- If Γ is a set of formulae, then Γ is absolute for $\mathcal{M}, \mathcal{N}$ iff φ is absolute for $\mathcal{M}, \mathcal{N}$.

Example.
- If $\mathcal{M} \subseteq \mathcal{N}$ both satisfy extensionality, then $\emptyset$ is absolute via the formula $\forall x \in a(x \neq x)$.
- $\mathcal{P}(2)$ is not absolute between 4 and V. In 4 it will not have $\{0,1,2,3\}$ for example.

Example. $\varphi \leftrightarrow \psi$ does not imply $\varphi^M \leftrightarrow \psi^M$. For example, let $\varphi(v) = \forall x(x \notin v)$ in ZF this defines $\emptyset$. Now the following expresses $0 \in z$:

$$\psi(z) \equiv \exists y(\varphi(y) \wedge y \in z)$$

$$\theta(z) \equiv \forall y(\varphi(y)) \to y \in z.$$

Note that $\exists! y\, \varphi(y)$, then these are equivalent. However $a = 0$, $b = \{0\}$, $c = \{\{\{0\}\}\}$, then let $\mathcal{M} = \{a, b, c\}$. Then $\varphi^{\mathcal{M}}(a)$ holds, so $\psi^M(b)$, but $\varphi^M(c)$ also holds, so $\theta^{\mathcal{M}}(b)$ fails.

The common theme amongst our counterexamples so far has been the lack of transitivity.

Definition 3.6 (Transitive Class)**.** Given classes $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is **transitive** in $\mathcal{N}$ if

$$\forall x, y\, (x, y \in N \wedge x \in M \wedge y \in x \to y \in \mathcal{M}).$$

4 The Levy Hierarchy

Definition 4.1 (Δ_0)**.** The class Δ_0 of formulae is the smallest class Γ which is closed under the following:

- Closure under atomic formulae:

  $$\forall i,\, j\, (v_i \in v_j) \in \Gamma,$$

  and

  $$(v_i = v_j) \in \Gamma.$$

- Closure under propositional connectives, i.e. if $\varphi, \psi \in \Gamma$, then $\varphi \vee \psi \in \Gamma$, $\neg\varphi \in \Gamma$.
- Closure under *bounded* quantifiers, i.e. for all i, j,

  $$\varphi \in \Gamma \Rightarrow (\forall v_i \in v_j \varphi) \in \Gamma, \quad \exists v_i \in v_j .\, \varphi \in \Gamma.$$

Example.

$v_1 \in v_2 \vee \forall v_3 \in v_4(v_4 = v_5) \to \exists v_1 \in v_1(v_1 = v_2) \in \Gamma$ (even though it's meaningless nonsense)

Then for the rest of the Levy Hierarchy, we proceed by induction to define:

Definition 4.2 (Levy Hierarchy)**.** We set $\Sigma_0 = \Pi_0 = \Delta_0$.

- If φ is Π_{n-1}, then $\exists v_i \varphi$ is Σ_n.
- If φ is Σ_{n-1}, then $\forall v_i \varphi$ is Π_n.

Example. $\forall v_1 \exists v_2 \forall v_3(v_4 = v_7)$ is Π_3. But $\forall v_1(v_1 = v_2) \wedge v_3 \in v_4$, or $\forall v_1(v_1 = v_2 \wedge \exists v_3(v_3 \in v_4))$ are not in Π_n or Σ_n for any n.

Definition 4.3 (The Levy Hierarchy for a theory)**.** Given a $\mathcal{L}$-theory $\mathcal{T}$, we let $\Sigma_n^{\mathcal{T}}$ be the class of formulae Γ such that for any $\varphi \in \Gamma$, there exists $\psi \in \Sigma_n$ such that $\mathcal{T} \vdash \varphi \leftrightarrow \psi$. $\Pi_n^{\mathcal{T}}$ is defined the same way.

Finally, we say that a formula is Δ_n^{T} if there is a ψ in Σ_n and a $\theta \in \Pi_n$ such that $\mathcal{T} \vdash \varphi \leftrightarrow \psi \wedge \varphi \leftrightarrow \theta$.

Lecture 3 **Warning:**

- $\exists x_1 \forall x_2 \exists x_3 \forall y(y \in v \to v \neq v)$ is Σ_4 as written, but it is clearly logically equivalent to $\forall y \in v(v \neq v)$ which is Σ_0.
- In Z, it can be the case that you have φ being Σ_1^{Z} but $\forall x \in a.\, \varphi$ is not Σ_1^{Z}.
- Δ_n only ever makes sense in the context of a theory $\mathcal{T}$ for $n > 0$.
- We only work in classical logic in this course, but in intuitionistic logic, these classes are very badly behaved, since you can have φ being $\Sigma_1^{\mathcal{T}}$ but $\neg\varphi$ not being $\Pi_1^{\mathcal{T}}$, and in fact these classes will not cover the whole universe.

Lemma 4.4.

- *If φ and ψ are in Σ_n^{ZF} then so are $\exists v_i \varphi$, $\varphi \wedge \psi, \varphi \vee \psi$, as well as $\exists v_i \in v_j \varphi$, and $\forall v_i \in v_j \varphi$.*
- *If φ is in Σ_n^{ZF}, then $\neg\varphi$ is in Π_n^{ZF}.*
- *For every φ there exists n such that φ is in Σ_n^{ZF}.*

- *If φ is in Σ_n^{ZF} and $m \geqslant n$ then φ is in Σ_m^{ZF}.*

Proof. Example sheet 1. □

4.1 Absoluteness of Δ_0

Theorem 4.5. *Suppose that $\mathcal{M}$ is transitive in $\mathcal{N}$ and $\varphi(\bar{u})$ is a Δ_0 formula, then for any $\bar{a} \in \mathcal{M}$,*

$$M \vDash \varphi(\bar{a}) \text{ iff } \mathcal{N} \vDash \varphi(\bar{a}).$$

Proof. By induction on the class Δ_0.

- Atomic formulas, propositional connectives are immediate, so we only care about bounded quantifiers (and the only hard one is existential, since the universal quantifier is also immediate).
- Suffices to consider $\exists x \in a.\, \varphi$ where φ is absolute between $\mathcal{M}$ and $\mathcal{N}$.
  - $\Rightarrow$ If $\mathcal{M} \vDash \exists x \in a.\, \varphi$, then by definition

    $$\mathcal{M} \vDash \exists x (x \in a \wedge \varphi(x)).$$

    Fix $b \in \mathcal{M}$ such that $\mathcal{M} \vDash b \in a \wedge \varphi(b)$. Then $a, b \in \mathcal{N}$. So $\mathcal{N} \vDash b \in a \wedge \varphi(b)$. I.e. $\mathcal{N} \vDash \exists x \in a.\, \varphi(x)$.
  - $\Leftarrow$ Suppose $\mathcal{N} \vDash \exists x \in a.\, \varphi(x)$, where $a \in M$. Then fix $b \in \mathcal{N}$ such that

    $$\mathcal{N} \vDash b \in a \wedge \varphi(b).$$

    Since $\mathcal{M}$ is transitive in $\mathcal{N}$, $b \in \mathcal{M}$. Therefore $\mathcal{M} \vDash b \in a \wedge \varphi(b)$, and so $\mathcal{M} \vDash \exists x \in a.\, \varphi(x)$, as required.

□

Proposition 4.6. *The following are Δ_0^{ZF} and therefore absolute between transitive models:*

1. $x \subseteq y$.
2. $a = \{x, y\}$.
3. $a = (x, y)$.
4. $a = x \times y$.
5. $a = \cup b$.

6. $\mathrm{Tc}(a)$, *the transitive closure of* a.
7. $x = \emptyset$.
8. r *is a relation.*
9. r *is a function.*
10. r *is a relation with domain* a *and range* b.
11. $r``a$, *where* $r``a := \{y : \exists x \in a. (x, y) \in r\}$.
12. $r \restriction a$.

Proof. I think this is laid out well in Jech. ☐

Remark. $\mathrm{cf}(a)$, "a is a cardinal", ω_1 and $y = \mathcal{P}(x)$ are *not* absolute.

Lemma 4.7. *"a is finite" is* Δ_1^{ZF}.

Proof. Example sheet 1. ☐

Proposition 4.8. *Suppose* $\mathcal{M} \subseteq \mathcal{N}$, *then*

- Σ_1 *formulae are upwards absolute.*
- Π_1 *formulae are downwards absolute.*

Proof. Easy, think about what "$\forall$" and "$\exists$" mean.

Corollary 4.9. Δ_1^{ZF} *formulae are absolute between transitive models.*

Proof. A Δ_1^{ZF} statement is Π_1^{ZF}, so downwards absolute, and Σ_1^{ZF}, so upwards absolute. ☐

Lemma 4.10 (ZF)**.** *The statement "α is an ordinal" is absolute.*

Proof. Exercise. α is an ordinal if it is a transitive set of transitive sets. The latter can be written as

$$\forall \beta \in \alpha \ \forall \gamma \in \beta \ (\gamma \in \alpha) \wedge \forall \beta \in \alpha \ \forall \gamma \in \beta \ \ \forall \delta \in \gamma (\delta \in \beta).$$

That wasn't much of an exercise. ☐

Lemma 4.11. *The statement "r is a strict total ordering of α" is Δ_0.*

Proof. We need to state

- r is transitive on a.
- r satisfies trichotomy (on a).
- r is irreflexive (on a).

These all use bounded quantifiers. ☐

Corollary 4.12. *The statement "x is a transitive set, totally ordered by $\in$" is Δ_0.*

Lemma 4.13 (ZF)**.** *The statement "r is a well ordering on a" is Δ_1^{ZF}.*

Proof.

- Π_1: We can say: r is a relation on a and,

$$\forall X(\exists z \in X)(z = z) \wedge X \subseteq a) \to \exists z \in X \, \forall y \in X \, (y, z) \notin r.$$

- For Σ_1, we first need the following claim: a relation is well-founded iff there exists a function

$$f : a \to \text{Ord} \text{ s.t. } (y, x) \in r \Rightarrow f(y) < f(x).$$

Suppose f is well-founded, then define $f : \alpha \to \text{Ord}$ by $f(x) = \sup\{f(y) + 1 : (y, x) \in r\}$.

- For the other direction, take $X \subseteq a$ non-empty, and look at $f"X \subseteq \text{Ord}$. This has a minimal element, which we'll call α. Then for any $z \in X$, if $f(z) = \alpha$, then

$$\forall y \in X. \, f(y) \geqslant \alpha.$$

So by definition $(y, z) \notin r$. (Not claiming that z is unique.)

Then to show Σ_1, note we have

$$\exists f(f \text{ is a function } \wedge \forall u \in \text{ran}(f)(a \in \text{Ord}) \wedge \forall x, y \in a((y, z) \in r \to f(y) \inf(x))).$$

☐

Proposition 4.14. *The following are* Δ_0^{ZF}*.*

- x *is a limit ordinal.*
- x *is a successor ordinal.*
- x *is a finite ordinal.*
- ω.
- $0, 1, 2, \ldots, 37, \ldots$.

□

Proposition 4.15. *The following are* Π_1^{ZF} *and hence downwards absolute:*

- κ *is a cardinal.*
- κ *is regular.*
- κ *is a limit cardinal.*
- κ *is a strong limit cardinal.*

5 Relativizing Axioms

Lemma 5.1 (ZF)**.** *Suppose* W *is a transitive class, and* $W \neq \emptyset$*. Then we have (Extensionality)*^{W}*, (Empty set)*^{W}*, (Foundation)*^{W}*.*

Proof.

- Extensionality says that
$$\forall x(\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)).$$
Then we want to relativise this to W, so we get:
$$\forall x \in W(\forall y \in W(\forall z \in W(z \in x \leftrightarrow z \in y)^W \rightarrow (x = y)^W)).$$
Then we note that the two formulae in the bracket are absolute, so they hold in W if and only if they *actually* hold. Then since W is transitive, if $x, y \in W$, then also $x, y \subseteq W$. Suppose $x \neq y$, then by extensionality in the universe, we can fix some $z \in x$, $z \notin y$. But then $z \in W$, so we have $\exists z \in W(z \in x \wedge z \notin y)$ which contradicts our assumption.

- Foundation is a very similar argument, write it all out, then use foundation in the background universe to show that foundation holds in the inner model.
- For empty set, it is even easier. We first fix $x \in W$ such that $x \cap W = \emptyset$ using the fact that W is non-empty, and that foundation holds in V. Since W is transitive, $x \subseteq W$. Therefore $x = \emptyset \in W$. Moreover $(x = \emptyset)$ is Δ_0, so it is absolute between transitive models.

Lecture 4

Lemma 5.2.

- *Suppose W is a transitive class, then if*

$$\forall x, y \in W,\ \{x, y\} \in W$$

then $(\text{Pairing})^W$ holds.

- *If for any $x \in W$, $\cup x \in W$, then $(\text{Union})^W$ holds.*
- *If $\omega \in W$, then $(\text{Infinity})^W$ holds.*
- *If for every φ, with $\mathrm{Fr}(\varphi) \subseteq \{x, a, v_1, \ldots, v_n\}$, we have that $\forall a, x_1, \ldots x_n \in W(\{x \in a | \varphi^W(x, a, v_1, \ldots, x_n)\} \in W)$, then separation must hold in W.*
- *If for every formula φ with $\mathrm{Fr}(\varphi) \subseteq \{x, z, a, v_1, \ldots, v_n\}$, we have that for any $a, v_1, \ldots, v_n \in W$. If*

$$\forall x \in a \exists! y \in W\ \varphi^W(x, y, a, v_1, \ldots, v_n).$$

we have that

$$\exists b \in W \left(\{y | \exists x \in a.\, \varphi^W(x, y, a, v_1, \ldots, v_n)\} \subseteq\right)$$

then replacement holds in W.

- *If $\forall a \in W$, $\exists b \in W$ such that $(\mathcal{P}(a) \cap W = b)$, then $(\text{Power Set})^W$ holds.*

Corollary 5.3. *If W is a transitive class satisfying the conditions of the previous lemma, then $(\mathsf{ZF})^W$ i.e. W is a model of ZF.*

6 Transfinite Recursion

Definition 6.1 (Set-like relation)**.** A relation R is **set-like** on a class A iff $\forall x \in A. \{y \in A \mid yRx\}$ is a set.

Example.

- $\in$ is set-like.
- Any relation on a set will be set-like.

Definition 6.2 (Absolute class)**.** Let A be a class and fix φ such that $A = \{x \mid \varphi(x)\}$. Then $A^W = \{x \mid \varphi^W(x)\}$. We will say that A is **absolute** if $A^W = A \cap W$.

Definition 6.3 (Absolute class-relation)**.** View a class relation $R \subseteq V \times V$ as a collection of ordered pairs $\{(x, y) \mid \psi(x, y)\}$. Then $R^W = \{(x, y) \mid \psi^W(x, y)\}$. Say R is **absolute** for W iff $R^W = R \cap (W \times W)$.

Observe that if R is a function, we can only refer to the function R^W if we first check $(\forall x \exists! y. \psi(x, y))^W$. If we relativise a function, we will assume that we have already checked this. In this case $R^W : W \to W$ and R is an absolute function for W iff $R^W = R \restriction W$.

Theorem 6.4 (Transfinite Recursion)**.** *Let R be a relation which is well-founded and set-like on a class A, and let*

$$F : A \times V \to V$$

be a function. Given $x \in A$, let $\mathrm{pred}(A, x, R) = \{y \in A \mid yRx\}$. *Then there is a unique function $G : A \to V$ such that*

$$\forall x \in A. G(x) = F(x, G \restriction \mathrm{pred}(A, x, R)).$$

Theorem 6.5 (Absoluteness of transfinite recursion)**.** *Let R be a relation which is well-founded and set-like on a class A. Let $F : A \times V \to V$ be a class function and let $G : A \to V$ be the unique function given by transfinite recursion. Let W be a transitive model of* ZF*, and suppose that*

1. *A and F are absolute between W and the universe V.*
2. *R is also absolute for W (R is set-like on $A)^W$.*

3. $\forall x \in W$, $\text{pred}(A, x, R) \subseteq W$.

Then G is absolute for W.

Proof. By absoluteness, we have that $A^W = A \cap W$, and $R^W = R \cap (W \times W)$. Therefore every non-empty subset of A^W has an R^W minimal element, so $(R \text{ is well-founded on } A)^W$.

So apply transfinite recurison in W to define a (unique) function

$$G^W : A^W \to W$$

such that

$$\forall x \in A^W . \, G^W(x) = F^W(x, G^W \restriction \text{pred}^W(A^W, x, R^W)).$$

To prove G is absolute, it suffices to show $G^W = G \restriction A^W$. We prove this by transfinite induction in W. Suppose that for all yRx, $G^W(y) = G(y)$, then we can write

$$\begin{aligned} G^W(x) &= F^W(x, G^W \restriction \text{pred}^W(A^W, x, R^w)) \\ &= F(x, G \restriction \text{pred}(A, x, R)) \end{aligned}$$

□

Corollary 6.6. *The following are absolute for transitive models of* ZFC*:*

- $\text{rank}(x)$.
- $\text{Tc}(x)$.
- *Ordinal arithmetic operations, $+$ and $\cdot$, since these are both defined via transfinite recursion.*

7 The Reflection Thoerem

Recall the Tarski-Vaught test from model theory, which says:

Theorem (Tarski-Vaught Test)**.** *Let $\mathcal{M} \subseteq \mathcal{N}$ as structures, with universes M, N. Then TFAE:*

(i) $\mathcal{M} \preceq \mathcal{N}$ *($\mathcal{M}$ is an elementary substructure of $\mathcal{N}$).*

(ii) For any formula $\varphi(v, \bar{w})$ and $\bar{a} \in M$, if there exists $b \in N$ such that

$$\mathcal{N} \vDash \varphi(b, \bar{a}),$$

then there exists $c \in M$ such that $\mathcal{N} \vDash \varphi(c, \bar{a})$.

Proof. In Model Theory. □

Definition 7.1 (Subformula closed)**.** A finite list of formulae

$$\bar{\varphi} = \varphi_1, \ldots, \varphi_n$$

is said to be **subformula closed** if every subformula of a formula is on the list.

Example. $\varphi_i = \exists x. \psi_1$ and $\varphi_j = \psi_2 \wedge \psi_3$, then ψ_1, ψ_2, ψ_3 also appear on the list.

Lemma 7.2. *Let $\bar{\varphi} = \varphi_1, \ldots, \varphi_n$ be a subformula closed list, and $W \subseteq Z$ are two non-empty classes, then TFAE:*

(i) $\bar{\varphi}$ are absolute for W, Z

(ii) Whenever φ_i is of the form $\exists x. \varphi_j(x, \bar{y})$ where the free variables of φ_j are contained in $\{x, \bar{y}\}$, then

$$\forall \bar{y} \in W \left(\exists x \in Z. \varphi_j(x, \bar{y}) \to \exists x \in W. \varphi_j(x, \bar{y})\right).$$

Proof.

(i) ⇒ (ii) Suppose $\varphi_i \equiv \exists x. \varphi_j(x, \bar{y})$, fix $\bar{y} \in W$. Then $\varphi_i^Z(\bar{y}) \equiv \exists x \in Z. \varphi_j^Z(x, \bar{y})$. So if $\exists x \in Z. \varphi_j^Z(x, \bar{y})$, then, by absoluteness, $\varphi_1^W(\bar{y})$ holds, i.e. $\exists x \in W. \varphi_j^W(x, \bar{y})$.

Now, $W \subseteq Z$, and absoluteness of φ means that $\exists x \in W. \varphi_j(x, \bar{y})$.

(ii) ⇒ (i) We will prove this by induction on the length of φ_i.

- Base case is when φ_i is atomic or of the form $\varphi_j \vee \varphi_k$, or $\varphi_i \equiv \neg\varphi_j$ are immediate.

- Suppose $\varphi_i \equiv \exists x \varphi_j(x, \bar{y})$, and fix $\bar{y} \in W$. Then

$$\begin{aligned}\varphi_i^Z(\bar{y}) &\Leftrightarrow \exists x \in Z.\, \varphi_j^Z(x, \bar{y}) \\ &\Leftrightarrow \exists x \in W.\, \varphi_j^Z(x, \bar{y}) && (\Rightarrow \text{ assumption}, \Leftarrow, W \subseteq Z) \\ &\Leftrightarrow \varphi_i^W(\bar{y})\end{aligned}$$

□

Lecture 5

Theorem 7.3 (Reflection Theorem)**.** *Let W be a non-empty class, and suppose there is a class function F_W such that for any $\alpha \in \mathrm{Ord}$, $F_W(\alpha) = W_\alpha \in V$. Assume that*

(i) If $\alpha < \beta$, then $W_\alpha \subseteq W_\beta$.

(ii) If λ is a limit ordinal, then $W_\lambda = \cup_{\alpha<\lambda} W_\alpha$

(iii) $W = \cup_{\alpha \in \mathrm{Ord}} W_\alpha$

Then for any finite collection of formulae $\bar{\varphi} = \varphi_1, \ldots, \varphi_n$. $\mathsf{ZF} \vdash \forall\alpha\exists\beta > \alpha(\beta$ is a limit ordinal $\wedge\, \bar{\varphi}$ are absolute for W_β, $W)$.

Proof. Let $\bar{\varphi}$ be a finite list of formulae, then WLOG, we can assume $\bar{\varphi}$ is sub-formula closed and there are no universal quantifications (i.e. we always phrase $\forall x.\, \psi$ as $\neg\exists x.\, \neg\psi$).

Then if $\varphi_i \equiv \exists x.\, \varphi_j(x, \bar{y})$ where $\bar{y}$ is a tuple of length k_i, we define

$$F_i : W^{k_i} \to \mathrm{Ord}$$

by

$$F_i(\bar{y}) = \begin{cases} 0 & \text{if } \neg\exists x \in W.\, \varphi_j^W(x, \bar{y}) \\ \eta & \text{when } \eta \text{ is the least ordinal such that } \exists x \in W_\eta\, \varphi_j^W(x, \bar{y}) \end{cases}$$

Write $G_i(\delta) = \sup\{F_i(\bar{y}) \mid \bar{y} \in W_\delta^{k_i}\}$. We have the following:

- If φ_i is not of the above form, then $G_i(\delta) = 0$ for all δ.
- Finally, let $K(\delta) = \max\{\delta + 1, G_1(\delta), \ldots, G_n(\delta)\}$.

Note that F_i work in an analogous way to Skolem functions, and that F_i is a well-defined function and, by replacement, since W_δ is a set, F_i"$W_\delta^{k_i} \in V$. G_i is monotone, so if $\delta < \delta'$, then $G_i(\delta) \leqslant G_i(\delta')$. Then we claim that

$$\forall\alpha\, \exists\beta > \alpha\, (\beta \text{ is a limit ordinal} \wedge \forall\delta < \beta, \forall i \leqslant n.\ G_i(\delta) < \beta).$$

To prove this, set $\lambda_0 = \alpha$ and $\lambda_{t+1} = K(\lambda_t)$. Then $\beta = \sup_{t\in\omega} \lambda_t$. If $\delta < \beta$, then $\delta < \lambda_t$ for some t, so $G_i(\delta) \leqslant G_i(\lambda_t) \leqslant K(\lambda_t) = \lambda_{t+1} < \beta$.

To complete the theorem, suppose $\varphi_i \equiv \exists x.\, \varphi_j(x, \bar{y})$. Now fix $\bar{y} \in W_\beta$, Now suppose that $\exists x \in W.\, \varphi_j^W(x, \bar{y})$, since β is a limit, $\bar{y} \in W_\gamma$ for some $\gamma < \beta$. Thus $0 < F_i(\bar{y}) \leqslant G_i(\gamma) < \beta$, so by construction $\exists x \in W_\beta.\, \varphi_j(x, \bar{y})$, hence $\bar{\varphi}$ is absolute for W_β, W. □

Corollary 7.4 (Montague-Lévy Reflection)**.** *For any finite list of formulae $\bar{\varphi}$:*

$$\mathsf{ZF} \vdash \forall\alpha\exists\beta > \alpha(\bar{\varphi} \textit{ are absolute for } V_\beta)$$

Warning: Reflection is a theorem scheme, so for any choice of $\varphi_1, \ldots, \varphi_n$, it is a theorem of ZF that $\varphi_1, \ldots, \varphi_n$ are absolute for some V_β. We do not and can not have $\mathsf{ZF} \vdash \forall\bar{\varphi}\forall\alpha\exists\beta > \alpha.\, (\bar{\varphi}$ is absolute for W_β, W.).

Also, $\bar{\varphi}$ is absolute for W_β, W does not imply that $(\bar{\varphi})^{W_\beta}$, i.e. it doesn't mean that they actually hold.

Remark 7.5. If $\bar{\varphi}$ is any finite list of axioms of ZF then there are arbitrarily large β such that $\bar{\varphi}$ holds in V_β. But, if β is a limit cardinal, then $V_\beta \vDash$ Z(+Choice), so we can restrict $\bar{\varphi}$ to instances of replacement.

Corollary 7.6 (ZF)**.** *Let $\mathcal{T} \supseteq \mathsf{ZF}$ be a set of axioms in $\mathcal{L}_\in$, and $\varphi_1, \ldots, \varphi_n$ a finite list of axioms from $\mathcal{T}$. Then*

$$\mathcal{T} \vdash \forall\alpha\exists\beta > \alpha.\, (\bigwedge_{i=1}^{n} \varphi_i)^V_\beta.$$

Corollary 7.7 (ZFC)**.** *Let W be a class, $\bar{\varphi}$ a finite list of formulae in $\mathcal{L}_\in$. Then*

$$\begin{aligned}\mathsf{ZFC} \vdash \forall x \subseteq W(\,&\mathrm{Trans}(x) \to \exists y.(x \subseteq y \wedge \mathrm{Trans}(y)\\ &\wedge \bar{\varphi} \textit{ are absolute for } y, W \wedge |y| \leqslant \max\{\omega, |x|\}))\end{aligned}$$

Taking $x = \omega$, $W = V$.

Corollary 7.8. *Let $\mathcal{T}$ be any set of sentences in $\mathcal{L}_\in$ such that $T \vdash \mathsf{ZFC}$, and let $\varphi_1, \ldots, \varphi_n \in \mathcal{T}$. Then*

$$\mathcal{T} \vdash \exists y. \left(\text{Trans}(y) \wedge |y| = \omega \wedge \left(\bigwedge_{i=1}^{n} \right)^{y} \right).$$

Corollary 7.9. *Let $\mathcal{T}$ be any consistent set of sentences in $\mathcal{L}_\in$ such that $\mathcal{T} \vdash \mathsf{ZF}$. Then $\mathcal{T}$ is not finitely axiomatizable. That is, for any finite set of sentences Γ in $\mathcal{L}_\in$ such that $\mathcal{T} \vdash \Gamma$ there exists a sentence φ such that $T \vdash \varphi$ but $\Gamma \nvdash \varphi$. This is only true for first-order theories, e.g. Gödel-Bernays set theory (second-order set theory) is finitely axiomatizable.*

Proof. Let $\varphi_1, \ldots, \varphi_n$ be a set of sentences such that

$$\mathcal{T} \vdash \bigwedge_{i=1}^{n} \varphi_i.$$

Suppose for a contradiction that from $\varphi_1, \ldots, \varphi_n$, one can prove every axiom of $\mathcal{T}$. Then by reflection $\mathcal{T} \vdash \forall \alpha. \exists \beta > \alpha. \left((\bigwedge_{i=1}^{n} \varphi_i)^{V_\beta} \leftrightarrow (\bigwedge_{i=1}^{n} \varphi_i) \right)$. Fix β_0 to be the least ordinal such that $\bigwedge_{i=1}^{n} \varphi_i^{V_{\beta_0}}$. Then all axioms of $\mathcal{T}$ hold in V_{β_0}. (I.e. $V_{\beta_0} \vDash \mathcal{T}$). Since $\mathcal{T}$ extends ZF, basic absoluteness results hold. So if $\alpha \in V_{\beta_0}$, then $V_\alpha^{V_\beta} - V_\alpha \wedge V_{\beta_0} = V_\alpha$. SO V_α is absolute for V_{β_0}. Since $\mathcal{T}$ proves $\exists \alpha \bigwedge_{i=1}^{n} \varphi_i^{V_\alpha}$. Since V_{β_0} satisfies every axiom of $\mathcal{T}$, this must be true in V_{β_0}. So $\exists \alpha < \beta_0. \bigwedge_{i=1}^{n} \varphi_i^{V_\alpha}$. Contradicting the minimality of β_0. □

Lecture 6

8 Cardinal Arithmetic

For this section, we will work in ZFC, since in ZF it's quite hard to define what a cardinal even is.

Definition 8.1 (Cardinality)**.** The **cardinality** of x, written $|x|$ is the least ordinal α such that there is a bijection between α and x.

Definition 8.2 (Cardinal operations)**.** Let κ and λ be cardinals, then

- $\kappa + \lambda = |\{0\} \times \kappa \cup \lambda \times \{1\}|$.

- $\kappa \cdot \lambda = |\kappa \times \lambda|$
- $\kappa^\lambda = |{}^\lambda\kappa| = |\{f \,:\, f : \lambda \to \kappa\}|$
- $\kappa^{<\lambda} = \sup\{\kappa^\alpha \,:\, \alpha \in \text{Card},\ \alpha < \lambda\}$.

Theorem 8.3 (Hessenberg)**.** *If κ and λ are infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.*

Lemma 8.4. *If κ, λ, μ are infinite cardinals, then*

- $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
- $\left(\kappa^\lambda\right)^\mu = \kappa^{\lambda\mu}$.

Definition 8.5 (Cofinal/Cofinality)**.** A map $f : \alpha \to \beta$ is **cofinal** if $\sup(\text{ran}(f)) = \beta$. The cofinality of a limit ordinal γ, written $\text{cf}(\gamma)$ is the least ordinal α such that there is a cofinal map $f : \alpha \to \gamma$.

Remark.

- $\text{cf}(\gamma) \leqslant \gamma$.
- $\omega = \text{cf}(\omega) = \text{cf}(\omega + \omega) = \text{cf}(\aleph_{omega})$.
- $\text{cf}(\gamma) \leqslant |\gamma|$.

Definition 8.6 (Singular/Regular ordinals)**.** A limit ordinal is **singular** if $\text{cf}(\gamma) < \gamma$. If $\text{cf}(\gamma) = \gamma$, then we call the ordinal **regular**.

Theorem 8.7 (L)**.** *et γ be a limit ordinal, then:*

- *If γ is regular, then γ is a cardinal.*
- *γ^+ is a regular cardinal. (The cardinal successor, not the ordinal successor.)*
- $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$.
- *$\aleph_\alpha$ is regular whenever $\alpha = 0$ or α is a successor ordinal.*
- *If λ is a limit ordinal, then $\text{cf}(\aleph_\lambda) = \text{cf}(\lambda)$.*

Theorem 8.8. *If κ is a regular cardinal, and $\mathcal{F}$ is a family of sets, with $|\mathcal{F}| < \kappa$ and $|X| < \kappa$ for all $X \in \mathcal{F}$, then it will be the case that $|\cup \mathcal{F}| < \kappa$.*

Proof. We prove this by induction on $|\mathcal{F}| = \gamma < \kappa$. Suppose that the claim holds for γ, and let $\mathcal{F} = \langle X_\alpha \,|\, \alpha < \gamma + 1\rangle$. Then

$$\begin{aligned} |\cup F| &= |\cup_{\alpha<\gamma} X_\alpha \cup X_\gamma \\ &= |\cup_{\alpha<\gamma} X_\alpha| + |X_\gamma| \\ &= \max\{|\cup_{\alpha<\gamma} X_\alpha|, |X_\gamma|\} \\ &< \kappa \end{aligned}$$

Now suppose γ is a limit, and the claim holds for all $\beta < \kappa$. Let $\mathcal{F} = \langle X_\alpha \,|\, \alpha < \gamma\rangle$. Define $g : \gamma \to \kappa$ by $g(\beta) = |\cup_{\alpha<\gamma} X_\alpha|$, since κ is regular, $g(\gamma) = |\cup \mathcal{F}| < \kappa$. □

.

Definition 8.9 (L)**.** et κ_i for $i \in I$ be an index sequence of cardinal numbers and let $\langle X_i \,| i \in I\rangle$ be a sequence of pairwise disjoint sets with $|X_i| = \kappa_i$ for all $i \in I$. Then the **cardinal sum** of $\langle \kappa_i \,|\, i \in I\rangle$ is $\sum_{i\in I} |X_i|$.

The **cardinal product** of $\langle \kappa_i \,|\, i \in I\rangle$ is

$$\prod_{i\in I} \kappa_i = |\prod_{i\in I} X_i|$$

where

$$|\prod_{i\in I} X_i = \{f \,|\, f \text{ is a function, } \operatorname{dom}(f) = I, \text{ and } \forall i \in I.\, f(i) \in X_i\}.$$

Theorem 8.10 (König)**.** *Let I be an indexing set and suppose that $\kappa_i < \lambda_i$ for all $i \in I$. Then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. Let $\langle B \,|\, i \in I\rangle$ be a sequence of disjoint sets with $|B_i| = \lambda_i$, and let $B = \prod_{i\in I} B_i$. It will suffice to show that if $\langle A_i \,|\, i \in I\rangle$ is a sequence of subsets of B such that for all $i \in I$, $|A_i| = \kappa_i$, then $\cup_{i\in I} A_i \neq B_i$.

Given such a sequence, let S_i be the projection of A_i onto its i^{th} co-ordinate,

$$S_i = \{f(i) \mid f \in A_i\}.$$

Then $S_i \subseteq B_i$ and

$$|S_i| \leqslant |A_i| = \kappa_i < \lambda_i = |B_i|.$$

So fix a $t_i \in B_i \setminus S_i$. Finally, define $g \in B$, $g(i) = t_i$. By construction $g \notin A_i$ for all i, so $g \in B \setminus \cup_{i\in I} A_i$. □

Corollary 8.11 (Cantor's Theorem)**.** *If $\kappa \geqslant 2$ and λ is infinite, then $\kappa^\lambda > \lambda$.*

Proof.

$$\begin{aligned} \lambda &= \sum_{\alpha<\lambda} 1 \\ &< \prod_{\alpha<\lambda} 2 && \text{(König)} \\ &= 2^\lambda \\ &\leqslant \kappa^\lambda. \end{aligned}$$

□

Corollary 8.12.

$$\mathrm{cf}(2^\lambda) > \lambda.$$

Proof. Let $f : \lambda \to 2^\lambda$, we will show that

$$|\cup f\text{"}\lambda| < 2^\lambda|.$$

Since for all $i \in I$, $f(i) < 2^\lambda$,

$$\begin{aligned} \cup f\text{"}\lambda &= \sum_{i<\lambda} f(i) \\ &< \prod_{i<\lambda} 2^\lambda \\ &= (2^\lambda)^\lambda \\ &= 2^{\lambda\cdot\lambda} \\ &= 2^\lambda \end{aligned}$$

□

Corollary 8.13. *$2^{\aleph_0} \neq \kappa$ for any κ of cofinality $\aleph_0$. (In partiular, $2^{\aleph_0} \neq \aleph_\omega$.)*

Corollary 8.14. *$\kappa^{\mathrm{cf}(\kappa)} > \kappa$ for every infinite cardinality κ.*

8.1 Cardinal Exponentiation

Definition 8.15 (Generalized Continuum Hypothesis)**.** The **GCH** says that fpr every cardinal κ, $2^\kappa = \kappa^+$, i.e. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Theorem 8.16. *Assume that GCH holds, and κ, λ are infinite cardinals. Then there are three cases for cardinal exponentiation*

(i) *$\kappa \leqslant \lambda$ means that $\kappa^\lambda = \lambda^+$.*

(ii) *If $\mathrm{cf}(\kappa) \leqslant \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.*

(iii) *If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.*

Without GCH, we know much much less. the following thorem is essentially the only ZFC-provable restriction for regular cardinals.

Theorem 8.17. *If κ, λ are cardinals:*

(i) *If $\kappa < \lambda$, then $2^\kappa \leqslant 2^\lambda$.*

(ii) $\mathrm{cf}(2^\kappa) > \kappa$.

(iii) *If κ is a limit cardinal, then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.*

Theorem 8.18. *Let κ, λ be infinite cardinals, then*

(i) *If $\kappa \leqslant \lambda$ then $k^\lambda = 2^\lambda$.*

(ii) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geqslant \kappa$, then $\kappa^\lambda = \mu^\lambda$.*

(iii) *If $\kappa < \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then we have two possibilites:*

  (a) *If $\mathrm{cf}(\kappa) > \lambda$ then $k^\lambda = \kappa$.*

  (b) *If $\mathrm{cf}(\kappa) \leqslant \lambda$ then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.*

Theorem 8.19 (Silver)**.** *Suppose that κ is singular,* $\mathrm{cf}(\kappa) > \aleph_0$*, and* $2^\alpha = \alpha^+$ *for all* $\alpha < \kappa$*, then* $2^\kappa = \kappa^+$*.*

This says, essentially, that GCH cannot first break at a cardinal of countable cofinality.

Remark. It is consistent (relative to large cardinals) to have $2^{\aleph_n} = \aleph_{n+1}$ for all $n \in \omega$, but $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Theorem 8.20 (Shelah)**.** *Suppose that* $2^{\aleph_n} < \aleph_\omega$ *for all* $n \in \omega$*, then* $2^{\aleph_\omega} < \aleph_{\omega_4}$*.*

There is a big open question: Can we improve this bound? Can we prove, as conjectured, that $2^{\aleph_\omega} < \aleph_{\omega_1}$.

Lecture 7

9 Constructibility

One of our aims is to prove that Con(ZF) $\Rightarrow$ Con($\mathsf{ZFC}+\mathsf{GCH}$). Constructibility will help us to do this.

Recall that $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. Gödel's idea was to restrict this to "nice" subsets.

Definition 9.1 (Definable set)**.** A set x is said to be **definable** over $(\mathcal{M}, \in)$ if there exists $a_1, \ldots, a_n \in \mathcal{M}$ and a formula φ such that

$$x = \{z \in \mathcal{M} \mid (\mathcal{M}, \in) \vDash \varphi(z, a_1, \ldots, a_n)\}.$$

We can also write:

$$\mathrm{Def}(\mathcal{M}) = \{x \in \mathcal{M} \mid x \text{ is definable over } \mathcal{M}\}$$

Some observations about this definition:

- $\mathcal{M} \in \mathrm{Def}(\mathcal{M})$.
- $\mathcal{M} \subseteq \mathrm{Def}(\mathcal{M}) \subseteq \mathcal{P}(\mathcal{M})$

However, this definition needs formalising as it stands.

One method to formalise: Code formulas by elements of V_ω using Gödel codes. One then uses Tarski's satisfaction relation to define a formula Sat such that

$$\mathrm{Sat}(\mathcal{M}, E, \ulcorner\varphi\urcorner, x_1, \ldots, x_n) \leftrightarrow (\mathcal{M}, E) \vDash \varphi(x_1, \ldots, x_n)$$

where $\ulcorner\varphi\urcorner$ is the Gödel code for φ. We won't use this definition though.

Another method to formalise:

Definition 9.2 (The Constructible Hierarchy)**.** Define L_α by transfinite recursion as:

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$
- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$ when λ is a limit.
- $L = \cup_{\alpha \in \mathrm{Ord}} L_\alpha$.

Lemma 9.3. *For any ordinals* α, β*:*

(i) $\beta < \alpha \to L_\beta \subseteq L_\alpha$.

(ii) $\beta < \alpha \to L_\beta \in L_\alpha$.

(iii) $\mathrm{Trans}(L_\alpha)$.

(iv) $\alpha = \mathrm{Ord} \cap L_\alpha$.

(v) $\mathrm{Trans}(L)$ *and* $\mathrm{Ord} \subseteq L$.

Definition 9.4 (Inner model)**.** Let $\mathcal{T}$ be a set of axioms of $\mathcal{L}_{in}$, and W be a class. W is called an **inner model** of $\mathcal{T}$ if:

(i) $\mathrm{Trans}(W)$

(ii) $\mathrm{Ord} \subseteq W$.

(iii) $(T)^W$, that is, for every formula φ in T, φ^W.

Theorem 9.5. *L is an inner model of* ZFC*.*

Proof.

- Extensionality and foundation: Since L is transitive, L satisfies these.

- Empty set: $\emptyset^L = \emptyset = L_0 \in L$.
- Pairing: Given $a, b \in L$, have to prove $\{a, b\} \in L$. Fix α such that $a, b \in L_\alpha$. Then $\{a, b\} = \{x \in \alpha : L_\alpha \vDash x = a \vee x = b\}$. So $\{a, b\} \in \mathrm{Def}(L_\alpha)$.
- Unions: Fix $a \in L_\alpha$. So $\cup\alpha \subseteq L_\alpha$. Then

$$\cup a = \{x \in L_\alpha \,|\, (L_\alpha, \in) \vDash \exists z. (z \in a \wedge x \in z)\} \in \mathrm{Def}(L_\alpha).$$

- Infinity: $\omega = \{n \in L_\omega \,|\, (L_\omega, \in) \vDash n \in \mathrm{Ord}\}$.

These are all the "easy" axioms. Now for the harder ones:

- Separation: Let φ be a formula, $a, \bar{u} \in L_\alpha$. Then we claim that $b = \{x \in a \,|\, \varphi^L(x, baru)\} \in L$. Using the reflection theorem, we can find $\beta > \alpha$ such that

$$\mathsf{ZF} \vdash \forall x \in L_\beta \left((\varphi)^L(x, \bar{u}) \leftrightarrow \varphi^L(x, \bar{u})\right)$$

Moreover, φ^{L_β} holds iff $(L_\beta, \in) \vDash \varphi(x, \bar{u})$. Therefore

$$\begin{aligned} &\{x \in a \,|\, \varphi^L(x, \bar{u})\} \\ &= \{x \in a \,|\, \varphi^{L_\beta}(x, \bar{u})\} \\ &= \{x \in L_\beta \,|\, (L_\beta, \in) \vDash \varphi(x, \bar{u})\} \in \mathrm{Def}(L_\beta). \end{aligned}$$

- Replacement: It suffices to prove that if $\alpha \in L$ and $f : \alpha \to L$ is a function, then there exists $\gamma \in \mathrm{Ord}$ such that $f"a \subseteq L_\gamma$. Observe that for every $x \in a$, there exists $\beta \in \mathrm{Ord}$ such that $f(x) \in L_\beta$. So by replacement in V, then there exists an ordinal γ such that for every $x \in a$, there exists $\beta \in \gamma$ such that $f(x) \in L_\beta$. Since $\beta \in \gamma \Rightarrow L_\beta \subseteq L_\gamma$, we have that $\forall x \in a. f(x) \in L_\gamma$.
- Power set: Suffices to prove that if $x \in L$, then $\mathcal{P}(x) \cap L \in L$. Take $x \in L$, then using replacement in V fix $\gamma \in \mathrm{Ord}$ such that $\mathcal{P}(x) \cap L \subseteq L_\gamma$. Then

$$\mathcal{P}(x) \cap L = \{z \in L_\gamma \,|\, (L_\gamma, \in) \vDash z \subseteq x\} \in \mathrm{Def}(L_\gamma)$$

□

9.1 Gödel functions

Note for clarity: $(a, b, c) = (a, (b, c))$.

Definition 9.6 (Gödel functions)**.**

- $\mathcal{F}_1(x, y) = \{x, y\}$
- $\mathcal{F}_2(x, y) = \cup x$

- $\mathcal{F}_3(x,y) = x \setminus y$
- $\mathcal{F}_4(x,y) = x \times y$
- $\mathcal{F}_5(x,y) = \mathrm{dom}(x) = \{\mathrm{first} z \,|\, z \in x,\ z \text{ an ordered pair}\}$
- $\mathcal{F}_6(x,y) = \mathrm{ran}(x) = \{\mathrm{second} z \,|\, z \in x,\ z \text{ an ordered pair}\}$
- $\mathcal{F}_7(x,y) = \{(u,v,w) \,|\, (u,v) \in x,\ w \in y\}$
- $\mathcal{F}_8(x,y) = \{(u,w,v) \,|\, (u,v) \in x\, w \in y\}$
- $\mathcal{F}_9(x,y) = \{(v,u) \in y \times x \,|\, u = v\}$
- $\mathcal{F}_{10}(x,y) = \{(v,u) \in y \times x \,|\, u \in v\}$.

Proposition 9.7. *The following can all be written as a finite combination of $\mathcal{F}_1$ to $\mathcal{F}_8$: $\{x\}$, $x \cup y$, $x \cap y$, (x,y), (x,y,z). For example $\{x\} = \mathcal{F}_1(x,x)$.*

Proposition 9.8. *For every $i \leqslant 10$, $z = \mathcal{F}_i(x,y)$ can be written using a Δ_0-formula.*

Lemma 9.9 (Gödel's normal form)**.** *For every Δ_0 formiula $\varphi(x_1,\dots,x_n)$ with $\mathrm{Fr}(\varphi) \subseteq \{x_1,\dots,x_n\}$, there is a term $\mathcal{F}_\varphi$ built from the symbols $\mathcal{F}_1 - \mathcal{F}_{10}$ such that*

$$\mathsf{ZF} \vdash \forall a_1,\dots,a_n.\, \mathcal{F}_\varphi(a_1,\dots,a_n) = \{(x_n,\dots,x_1) \in a_n \times \cdots \times a_1 \,|\, \varphi(x_1,\dots,x_n)\}$$

Remark.

- We reverse the ordering in the Gödel normal form for technical reasons of being easier to work with.
-
  - $\mathcal{F}_2$ will correspond to $\vee$
  - $x \cap y$ will correspond to $\wedge$
  - $\mathcal{F}_3$ will correspond to $\neg$
  - $\mathcal{F}_9, \mathcal{F}_{10}$ are atomic.

$\mathcal{F}_7$, $\mathcal{F}_8$ deal with ordered n-tuples. (x_1,x_2,x_3) formed using x_1 and (x_2,x_3). It canno be formed using (x_1,x_2) and x_3, so we have $\mathcal{F}_7$ and $\mathcal{F}_8$.

Definition 9.10 (Closed under Gödel functions)**.** A class C is **closed under Gödel functions** iff

$$\mathcal{F}_i(x, y) \in C \text{ whenever } x, y \in C.$$

Given a set b, we write $\mathrm{cl}(b)$ for the smallest set which contains b as subset and which is closed under Gödel functions.

Definition 9.11. Let b be a set, define $\mathcal{D}^n(b)$ inductively.

- $\mathcal{D}^0(b) = b$.
- $\mathcal{D}^{n+1}(b) = \{\mathcal{F}_i(x, y) \mid x, y \in \mathcal{D}^n(b),\, i \leqslant 10\}$.

Note that $\mathrm{cl}(b) = \cup_{n\in\omega}\mathcal{D}^n(b)$.

Lemma 9.12. *If $\mathcal{M}$ is a transitive class, closed under Gödel functions, then $\mathcal{M}$ satisfies Δ_0-separation.*

Proof. Let $\varphi(x_1, \dots, x_n)$ be Δ_0, and let $a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n \in \mathcal{M}$.

Let $Y = \{x_i \in a \mid \varphi(b_1, \dots, b_{i-1}, x_i b_{i+1}, \dots, b_n)\}$. Let $\mathcal{F}_\varphi$ be the formula from Gödel's normal form, then for any $c_1, \dots, c_n \in \mathcal{M}$:

$$F_\varphi(c_1, \dots, c_n) = \{(x_n, \dots, x_1) \in c_n \times \cdots \times c_1 \mid \varphi(x_1, dots, x_n)\} \in \mathcal{M}.$$

So, since $\{b_j\} = \mathcal{F}_1(b_j, b_j) \in \mathcal{M}$, $\mathcal{F}_\varphi(\{b_1\}, \dots, \{b_{i-1}\}, a, \{b_{i+1}\}, \dots, \{b_n\}) \in \mathcal{M}$. Then you can show $Y \in \mathcal{M}$ by taking $\mathcal{F}_6$(ran) $n - i$ times, and then taking $\mathcal{F}_5$ (dom). □

Theorem 9.13. *For every transitive set $\mathcal{M}$,*

$$\mathrm{Def}(\mathcal{M}) = \mathrm{cl}(\mathcal{M} \cup \{\mathcal{M}\}) \cap \mathcal{P}(\mathcal{M})$$

.

Lecture 8

Proof. ($\subseteq$): Let φ be a formula. Then $\varphi^{\mathcal{M}}$ is Δ_0. Therefore, there is a term $\mathcal{G}$ built from $\mathcal{F}_1, \ldots, \mathcal{F}_{10}$ such that for $a_1, \ldots, a_n \in \mathcal{M}$:

$$\begin{aligned}\{x \in \mathcal{M} \mid (\mathcal{M}, \in) \models \varphi(x, a_1, \ldots, a_n)\} &= \{x \in \mathcal{M} \mid \varphi^{\mathcal{M}}(x, a_1, \ldots, a_n)\} \\ &= \mathcal{G}(\mathcal{M}, a_1, \ldots, a_n) \\ &\in \mathrm{cl}(M \cup \{M\})\end{aligned}$$

($\supseteq$): First, we have the following claim:

Claim: If $\mathcal{G}$ is built from $\mathcal{F}_1, \ldots, \mathcal{F}_{10}$, then for any $x, a_1, \ldots, a_n$, the statements

$$x = \mathcal{G}(a_1, \ldots, a_n) \qquad\qquad x \in \mathcal{G}(a_1, \ldots, a_n)$$

are Δ_0. To prove this, suppose that $X, Y \in \mathcal{D}^k(a_1, \ldots, a_n)$. Then

$$x = \mathcal{F}_1(X, Y) \leftrightarrow \forall z \in x(z = X \vee z = Y) \wedge \exists w \in x(w = X) \wedge \exists w' \in Y(w' = Y)$$

$x \in \mathcal{F}_1(X, Y) \leftrightarrow x \in \{X, Y\} \leftrightarrow x = X \vee x = Y$. Similar arguments work for $i \in \{2, 3, \ldots, 10\}$.

Assuming the claim, let $Z \in \mathrm{cl}(M \cup \{M\}) \cap \mathcal{P}(M)$. Fix $\mathcal{G}$ to be a term built from $\mathcal{F}_1, \ldots, \mathcal{F}_{10}$ such that

$$Z = \mathcal{G}(\mathcal{M}, a_1, \ldots, a_n).$$

Let p bea Δ_0 formula such that

$$x \in \mathcal{G}(\mathcal{M}, a_1, \ldots, a_n) \text{ iff } \varphi(x, \mathcal{M}, a_1, \ldots, a_{10}).$$

Then $\mathcal{G}(\mathcal{M}, a_1, \ldots a_n) = \{x \in M \mid \varphi(x, \mathcal{M}, a_1, \ldots, a_n)\}$. We need a formula ψ such that

$$\psi^{\mathcal{M}} \leftrightarrow \varphi(x, \mathcal{M}, a_1, \ldots, a_n).$$

For example, define ψ from φ by the following replacements:

(i) $\exists v_i \in \mathcal{M}$ is replaced by $\exists v_i$

(ii) $\forall v_i \in \mathcal{M}$ is replaced by $\forall v_i$.

(iii) $v_i \in \mathcal{M}$ replaced by $v_i = v_i$.

(iv) $\mathcal{M} = \mathcal{M}$ replaced by $v_0 = v_0$

(v) $\mathcal{M} \in \mathcal{M}$, $\mathcal{M} \in v_i$, $M = v_i$ replaced by $\neg(v_0 = v_0)$.

Then

$$\begin{aligned}Z &= \mathcal{G}(\mathcal{M}, a_1, \ldots, a_n) \\ &= \{x \in \mathcal{M} \mid (\mathcal{M}, \in) \models \psi(x, a_1, \ldots, a_n)\} \in \mathrm{Def}(\mathcal{M}).\end{aligned}$$

□

Now, recall lemma 8.9, that there is a Gödel normal form for all Δ_0 functions. We will prove this now:

Proof. We will call a formula φ a **termed-formula** (or a t-formula) if the conclusoon of the lemma holds for φ. We will only use the logical symbols $\vee, \wedge, \neg, \exists$. The only occurrence of existentials will be in formulas of the form:

$$\varphi(x_1, \ldots, x_n) \equiv \exists x_{m+1} \in x_j \psi(x_1, \ldots, x_{m+1})$$

where $j \leqslant m \leqslant n$. For example:

$$\varphi(x_1, x_2, x_3, x_4) \equiv \exists x_3 \in x_1(x_1 \in x_2 \wedge x_3 = x_1)$$

is allowed, but

$$\exists x_1 \in x_2 \psi$$

is not, since the containment of elements is going up, and

$$\exists x_3 \in x_1(x_3 \in x_2 \vee \exists x_4 \in x_1 \psi)$$

is not because ψ needs to be a statement of at most 3 variables.

Note that every Δ_0 formula is equivalent to one satisfying these assumptions. We allow for dummy variables, so $\varphi(x_1, x_2) = x_1 \in x_2$ and $\varphi(x_1, x_2, x_3) = x_1 \in x_2$ are distinct. There are four sections of the proof

1. Logical points
2. Propositional connectives
3. Atomic formulas
4. Existentially Bounded

Section 1: Logical Points

(a) If $\mathsf{ZF} \vdash \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ and φ is a t-formula, then ψ is a t-formula, since

$$\mathcal{F}_\psi(\bar{a}) = \{\bar{x} \in \bar{a} \,|\, \varphi(\bar{x})\} = \mathcal{F}_\varphi(\bar{a})$$

(b) For all m, n if $\varphi(x_1, \ldots, x_n) \equiv \psi(x_1, dots, x_m)$ and ψ is a t-formula, then so is ψ. There are a few cases:

  (i) $n \geqslant m$. Prove by induction on n. If $n = m$ then trivial. $\varphi(x_1, \ldots, x_{n+1}) \equiv \psi(x_1, \ldots, x_m)$ then $\varphi(x_1, \ldots, x_{n+1}) \equiv \theta(x_1, \ldots, x_n)$ when θ is a t-formula. Then

  $$\mathcal{F}_\varphi(a_1, \ldots, a_n, a_{n+1}) = a_{n+1} \times \mathcal{F}_\theta(a_1, \ldots, a_n) = \mathcal{F}_4(a_n, \mathcal{F}_\theta(a_1, \ldots, a_n)).$$

  This is why we reverse the ordering, to make this section easier.

(ii) If $n \leqslant m$, also by induction. $n = m$ is similarly trivial. If

$$\varphi(x_1, \ldots, x_{n-1}) \equiv \psi(x_1, \ldots, x_m)$$

then

$$\varphi(x_1, \ldots, x_{n-1}) \equiv \theta(x_1, \ldots, x_n)$$

and $\{0\} = \{\mathcal{F}_3(a_1, a_1)\} = \mathcal{F}_1(\mathcal{F}_3(a+_1, a_1), \mathcal{F}_3(a_1, a_1))$. Then

$$\begin{aligned}\mathcal{F}_\varphi(a_1, \ldots, a_{n-1}) &= \{(x_{n-1}, \ldots, x_1) \in a_{n-1} \times \cdots \times a_1 | \varphi(x1, \ldots, x_{n-1})\} \\ &= \operatorname{ran}(\{(0, x_{n-1}, \ldots, x_1) \in \{0\} \times a_{n-1} \times \cdots \times a_1 \,|\, \theta(x_1, \ldots, x_{n-1}, 0)\}) \\ &= \mathcal{F}_6(\mathcal{F}_\theta(a_1, \ldots, a_{n-1}, \{0\}), a_1)\end{aligned}$$

and $\{0\}$ can be written using Gödel functions so we're done.

(c) If $\psi(x_1, \ldots, x_n)$ is a t-formula, and $\varphi(x_1, \ldots, x_{n+1}) \equiv \psi(x_1, \ldots, x_{n-1}, x_{n+1} \setminus x_n)$ then φ is a t-formula.

Firstly, if $n = 1$, then $\psi(x_1)$ is a t-formula. Consider $\psi(x_2 \setminus x_1)$. Then

$$\begin{aligned}\mathcal{F}_\varphi(a_1, a_2) &= \{(x_2, x_1) \in a_2 \times a_1 \,|\, \psi(x_2)\} \\ &= \{(x_2, x_1) \,|\, x_1 \in_1 \wedge x_2 \in \mathcal{F}_\psi(a_2)\} \\ &= \mathcal{F}_\psi(a_2) \times a_1 \\ &= \mathcal{F}_4(\mathcal{F}_\psi(a_2), a_1).\end{aligned}$$

Then if $n > 1$

$$\begin{aligned}\mathcal{F}_\varphi(a_1, \ldots, a_{n+1}) = \{&(x_{n+1}, \ldots, x_1) \in a_{n+1} \times \ldots a_1 \,| \\ &x_n \in a_n \wedge (x_{n+1}, x_{n-1}, \ldots, x_1) \in \mathcal{F}_4(a_1, \ldots, a_{n-1}, a_{n+1})\}\end{aligned}$$

$$= \mathcal{F}_8(\mathcal{F}_4(a_1, \ldots, a_{n-1}, a_{n+1}),_n)$$

(d) If $\psi(x_1, x_2)$ is a t-formula, and $\varphi(x_1, \ldots, x_n) \equiv \psi(x_{n-1} \setminus x_1, x_n \setminus x_2)$ then φ is a t-formula. Assume that $n > 2$, then

$$\begin{aligned}\mathcal{F}_\varphi(a_1, \ldots, a_n) &= \{(x_n, \ldots, x_1) \in a_n \times \cdots \times a_1 \,|\, (x_n, x_{n-1}) \in \mathcal{F}_\psi(a_{n-1}, a_n)\} \\ &= \mathcal{F}_7(\mathcal{F}_\psi(a_{n-1}, a_n), a_{n-2} \times \cdots \times a_1)\end{aligned}$$

Section 2: Propsitional Connectives

(e) If φ is a t-formula, so is $\neg\varphi$:

$$\mathcal{F}_{\neg\varphi}(a_1, \ldots, a_n) = (a_n \times \cdots \times a_1) \setminus \mathcal{F}_\varphi(a_1, \ldots, a_n).$$

(f) φ, ψ are t-formulas, then so is $\varphi \vee \psi$:

$$\mathcal{F}_{\varphi \vee \psi}(\bar{a}) = \mathcal{F}_\varphi(\bar{a}) \cup \mathcal{F}_\psi(\bar{a})$$

(g) φ, ψ are t-formulas, then so is $\varphi \wedge \psi$. Same as above but intersection instead of union.

Section 3: Atomic formulas

Lecture 9 (h) The formula $\varphi(x_1, \ldots, x_n) \equiv x_i = x_j$ is a t-formula for all $i, j \leqslant n$. There are n cases:

(i) $i = 1$, $j = 2$. Then:

$$\mathcal{F}_9(a_1, a_2) = \{(x_2, x_1) \in a_2 \times a_1 \mid x_1 = x_2\}$$

So $\mathcal{F}_\varphi$ isa formula using $\mathcal{F}_9$ and (b) (which lets us add dummy variables).

(ii) $j \geqslant i$. By induction. If $i = j$, then

$$\begin{aligned}\mathcal{F}_\varphi(a_1, \ldots, a_n) &= a_n \times \cdots \times a_1 \\ &= \{(x_n, \ldots, x_1) \in a_n \times \cdots \times a_1 \mid x_1 = x_1\}\end{aligned}$$

Suppose $j = i+1$, then let $\theta(x_1, \ldots, x_{i+1}) = (x_1 = x_2)[x_i/x_1, x_{i+1}/x_2]$. This is a t-formula by case (i) and (d), substitution. Then we get $\mathcal{F}_\varphi$ by adding dummy variables.

Then for the general case, $\varphi \equiv x_i = x_{j+1}$. By (c), $\varphi(x_1, \ldots, x_{j+1}) \equiv (x_i = x_j)[x_{j+1}/x_j]$ is a t-formula.

(iii) If $i > j$, then $x_i = x_j$ is logically equivalent to $x_j = x_i$ which is a t-formula by case (ii).

(i) The statement $\varphi(x_1, \ldots, x_n) = x_i \in x_j$ is a t-formula for all $i, j \leqslant n$. Then $\mathcal{F}_\varphi$ is formed using $\mathcal{F}_{10}$ and (b) in the same way as in case (h).

If $\varphi \equiv x_i \in x_i$, then $\mathcal{F}_\varphi(a_1, \ldots, a_n) = \emptyset = a_1 \setminus a_1$.

For the general case,

$$\psi(x_1, \ldots, x_{n+2}) \equiv (x_i = x_{n+1}) \wedge (x_j = x_{n+2}) \wedge (x_{n+1} \in x_{n+2})$$

Then we have that $x_{n+1} \in x_{n+2}$ is a t-formula, since it is given by $(x_1 \in x_2)[x_{n+1}/x_1,\ x_{n+2}/x_2]$. So ψ is a t-formula. Then

$$\begin{aligned}\mathcal{F}_\varphi &= \operatorname{ran}(\operatorname{ran}(\{x_{n+2}, \ldots, x_1 \in a_j \times a_i \times a_n \times \cdots \times a_1 \mid x_i = x_{n+1, x_j = x_{n+2}, x_i \in x_j}\})) \\ &= \mathcal{F}_6(\mathcal{F}_6(\mathcal{F}_\psi(a_1, \ldots, a_n, a_i, a_j), a_1), a_1)\end{aligned}$$

Section 4: Bounded quantifiers Recall that the only occurrence of $\exists$ will be in formulas of the form:

$$\varphi(x_1, \ldots, x_n) \equiv \exists x_{m+1} \in x_1.\, \psi(x_1, \ldots, x_{m+1}),$$

where $j \leqslant m \leqslant n$.

(j) If $\psi(x_1, \ldots, x_{n+1})$ is a t-formula, then so is $\varphi(x_1, \ldots, x_n) \equiv \exists x_{n+1} \in x_j \,.\, \psi(x_1, \ldots, x_{n+1})$. Let $\theta(x_1, \ldots, x_{n+1}) \equiv x_{n+1} \in x_1$. Then $\theta \wedge \psi$ is a t-formula. Now

$$\begin{aligned} \mathcal{F}_{\theta\wedge\psi}(a_1, \ldots, a_n, \mathcal{F}_2(a_j, a_j)) &= \mathcal{F}_{\theta\wedge\psi}(a_1, \ldots, a_n, \cup a_j) \\ &= \{(x_{n+1}, \ldots, x_1) \mid x_{n+1} \in x_j, \, \forall k \leqslant n(x_k \in a_k), \, \psi(x_1, \ldots, x_{n+1})\} \end{aligned}$$

Then

$$\begin{aligned} \operatorname{ran}(\mathcal{F}_{\theta\wedge\psi}) &= \{(x_n, \ldots, x_1) \in a_n \times \cdots \times a_1 \mid \exists u.\, (u, x_n, \ldots, x_1) \in \mathcal{F}_{\theta\wedge\psi}(a_1, \ldots, a_n, \cup a_j)\} \\ &= \{(x_n, \ldots, x_1) \in a_n \times \cdots \times a_1 \mid \exists x_{n+1} \in x_j \,.\, \psi(x_1, \ldots, x_{n+1})\}. \end{aligned}$$

□

10 Axiom of constructibility

Definition 10.1 (Axiom of constructibility)**.** The **axiom of constructibility** is the assertion "$V = L$" which is equivalent to $\forall x \exists \alpha \in \mathrm{Ord}\,.\,(x \in L_\alpha)$.

Our aim is to show that being constructible is absolute.

Lemma 10.2. *The statement* $Z = \mathrm{cl}(M)$ *is* Δ_1^{ZF}.

Proof. It is Π_1 from its definition as the smallest set closed under Gödel functions: this comes from the statement

$$\forall W((M \cup \{M\} \subseteq W \wedge (\forall x, y \in W.\ \bigwedge_{i \leqslant 10} \mathcal{F}_i(x, y) \in W)) \rightarrow Z \subset W)$$

Then the statement is also Σ_1 from the inductive definition using $\mathcal{D}^n$. The statement goes like:

$$\exists W.\,(\mathrm{func}(W) \wedge \mathrm{dom}(W) = W \wedge Z = \cup\,\mathrm{ran}(W) \wedge W(0) = M \wedge W(n) \subseteq W(n+1) \wedge (\forall x, y \in W(n).\ \wedge_{i \leqslant 10} \mathcal{F}_i(x$$

So W is the function that maps n to $\mathcal{D}^n$, and we set Z to be the union of the ranges of these functions. □

Lemma 10.3. *The function* $\alpha \to L_\alpha$ *is absolute between transitive models of* ZF*.*

Proof. Define $G : \mathrm{Ord} \times V \to V$ by

$$G(\alpha, x) = \begin{cases} \mathrm{cl}(x(\beta) \cup \{x(\beta)\}), & \text{if } \alpha = \beta + 1,\ x \text{ a function with domain } \beta, \\ \cup_{\beta \in \alpha} x(\beta), & \alpha \text{ a limit}, \\ \emptyset & \text{o.w.} \end{cases}$$

Then G is an absolute function. By transfinite recursion, we have a function F such that

$$\begin{aligned} F &: \mathrm{Ord} \to V \\ F(\alpha) &\mapsto G(x, F \restriction \alpha) \end{aligned}$$

is absolute. Finally, $F(\alpha) = L_\alpha$ for all α. ☐

Theorem 10.4.

(i) L satisfies the axiom of constructibility.

(ii) L is the smallest inner model of ZF*. I.e., if M is an inner model, then $L \subseteq M$.*

Proof.

(i) Need to show that $(\forall x, \exists \alpha \in \mathrm{Ord}\,.\,(x \in L_\alpha))^L$. That is, $\forall x \in L, \exists \alpha \in \mathrm{Ord}\,.\,(x \in (L_\alpha)^L)$. Since the L_α hierarchy is absolute:

$$\forall x.\, (x \in L_\alpha \leftrightarrow x \in (L_\alpha)^L)$$

Since L contains every ordinal, if $x \in L$ then $x \in L_\alpha$ for some α. Thus $x \in (L_\alpha)^L$. So $L \vDash \alpha \in L \wedge x \in L_\alpha$.

(ii) Let M be an inner model of ZF. We construct L inside M, L^M. Then by absoluteness, for every $\alpha \in M \cap \mathrm{Ord}$, $L_\alpha = (L_\alpha)^M$. Thus $L_\alpha \subseteq M$ for every $\alpha \in M \cap \mathrm{Ord} = V \cap \mathrm{Ord}$. So $L \subseteq M$.

☐

11 The axiom of choice in L

Our aim is to show that a **strong** version of the axiom of choice holds in L. There is a (definable) global well-order. The idea is to define partial well-orderings $<_\alpha$ on L_α such that $<_{\alpha+1}$ **end-extends** $<_\alpha$. That is, if $y \in L_\alpha$ and $x \in L_{\alpha+1} \setminus L_\alpha$, then $y <_{\alpha+1} x$. Then we will take $<_L = \cup_\alpha <_\alpha$.

Theorem 11.1. *There exists a well-ordering of the class L.*

Proof. For each $\alpha \in \text{Ord}$ we will construct a well-ordering $<_\alpha$ on L_α such that if $\alpha < \beta$, then

(i) If $x <_\alpha y$, then $x <_\beta y$.

(ii) If $x \in L_\alpha$ and $y \in L_\beta \setminus L_\alpha$ then $x <_\beta y$.

In limit cases, we take unions. $<_\lambda = \cup_{\alpha<\lambda} <_\alpha$.

Then the difficult part is just to define $<_{\alpha+1}$. The idea is to take the ordering on L_α, then add $\{L_\alpha\}$ to the end oft his ordring, followed by elements of $\mathcal{D}(L_\alpha \cup \{L_\alpha\}) \setminus L_\alpha \cup \{L_\alpha\}$. Then by elements of $\mathcal{D}^2(L_\alpha \cup \{L_\alpha\}) \dots$.

The details of how we do this are as follows, define $<^n_{\alpha+1}$ by:

(1) $<^0_{\alpha+1}$ is the well ordering of $L_\alpha \cup \{L_\alpha\}$ that end-extends $<_\alpha$ by adding $\{L_\alpha\}$ as a max element.

(2) Suppose $<^n_{\alpha+1}$ is defined. Then $<^{n+1}_{\alpha+1}$ is the well-ordering of $\mathcal{D}^{n+1}(L_\alpha)$ satisfying: $x <^n_{\alpha+1} y$ iff $x <^n_{\alpha+1} y$ or $x \in \mathcal{D}^n(L_\alpha)$ and $y \in \mathcal{D}^{n+1}(L_\alpha) \setminus \mathcal{D}^n(L_\alpha)$, or $x, y \notin \mathcal{D}^n(L_\alpha)$ and (we choose an ordering on $\mathcal{D}^{n+1}(L_\alpha)$):

   (a) The least $i \leqslant 10$ such that $\exists u, v \in \mathcal{D}^n(L_\alpha)$ such that $x = \mathcal{F}_i(u, v)$ is less than the least such j such that $y = \mathcal{F}_j(u, v)$.

   (b) The least such i is equal to the least such j, and the $<^n_{\alpha+1}$-least $u \in \mathcal{D}^n(L_\alpha)$ such that $\exists v \in \mathcal{D}^n(L_\alpha)(x = \mathcal{F}_i(u, v))$ is less than the $<^n_{\alpha+1}$-least $u' \in \mathcal{D}^n(L_\alpha)$ such that $\exists v' \in \mathcal{D}^n(L_\alpha)$ such that $y = \mathcal{F}_i(u', v')$.

   (c) The least such i is equal to the least such j, and the $<^n_{\alpha+1}$-least such u's are equal and the least $v \in \mathcal{D}^n(L_\alpha)$... check moodle later today.

□

Remark. Just to give the main idea of the well-order of L, we assume that $<_\alpha$ is a well-order of L_α, then we need to well-order $L_{\alpha+1}$, end-extending $<_\alpha$, and it only remains to figure out the order of $\text{Def}(L_\alpha) \setminus L_\alpha \cup \{L_\alpha\}$. For this, we can well order $\mathcal{D}^{n+1}(L_\alpha \cup \{L_\alpha\}) \setminus \mathcal{D}(^n(L_\alpha \cup \{L_\alpha\})$, (and we start off by setting $D^0 = \{L_\alpha\}$ to be the least element in $<_{\alpha+1}$ which is not in L_α). Then for $x = \mathcal{F}_i(u, v)$ and $y = \mathcal{F}_j(u', v')$ in D^{n+1}, we order according to the least such Gödel functions that give the ordering. First we check if $i < j$, then we check if $u < u'$, tjen we check if $v < v'$, and otherwise $x = y$.

Lemma 11.2. *The relation $<_L$ is Σ_1-definable. Moreover, for every limit ordinal δ and $y \in L_\delta$, we have $x <_L y$ iff $x \in L_\delta$ and $L_\delta \vDash x <_L y$.*

Proof. Example sheet 2. □

Moreover, this gives the axiom of choice, because given $x \in L$, $<_L \restriction x$ gives a well-order of x.

12 GCH in L

Lemma 12.1 (ZFC)**.**

(i) *For all* $n \in \omega$, $L_n = V_n$.

(ii) *If* $\mathcal{M}$ *is infinite,* $|\mathcal{M}| = |\operatorname{Def}(\mathcal{M})|$,

(iii) *If* α *is an infinite ordinal,* $|L_\alpha| = |\alpha|$.

Proof. These are relatively obvious. □

Lemma 12.2 (Gödel's condensation lemma)**.** *For every limit ordinal* δ*, if* $(\mathcal{M}, \in) \preceq (L_\delta, \in)$*, then there exists some* $\beta \leqslant \delta$ *such that* $(\mathcal{M}, \in) \equiv (L_\beta, \in)$.

Proof. Let $\pi : (\mathcal{M}, \in) \to (\mathcal{N}, \in)$ be the Mostowski collapse of $\mathcal{M}$, and set $\beta = \mathcal{N} \cap \operatorname{Ord}$. Since $\mathcal{N}$ is transtitive, $\beta \in \operatorname{Ord}$. We shall prove that $\beta \leqslant \delta$, and $\mathcal{N} = L_\beta$.

- To prove $\beta \leqslant \delta$, suppose for a contradiction that $\delta < \beta$. Then $\delta \in \mathcal{N}$. So $\pi^{-1}(\delta) \in \mathcal{M}$. Now, since being an ordinal is absolute between transitive sets, $\mathcal{N} \vDash \delta \in \operatorname{Ord}$. Thus, $\mathcal{M} \vDash \pi^{-1}(\delta) \in \operatorname{Ord}$ (but we cannot say right now that $\pi^{-1}(\delta)$ is in fact an ordinal, since $\mathcal{M}$ isn't transtitive). But $\mathcal{M} \preceq L_\delta$, so $L_\delta \vDash \pi^{-1}(\delta) \in \operatorname{Ord}$. Since L_δ is transitive, $\pi^{-1}(\delta) \in \operatorname{Ord} \cap V$.

  Also, $\mathcal{M} \vDash x \in \pi^{-1}(\delta) \Leftrightarrow \mathcal{N} \vDash \pi(x) \in \delta$, since $\pi : (\pi^{-1}(\delta) \cap \mathcal{M}) \to \delta$ is an isomorphism. Therefore the order type, $\operatorname{otp}(\pi^{-1}(\delta) \cap \mathcal{M}) = \delta$. Then let $f : \delta \to \pi^{-1}(\delta) \cap \mathcal{M}$ be a strictly increasing enumeration. So for any $\alpha \in \delta$,
  $$\alpha \leqslant f(\alpha) < \pi^{-1}(\delta).$$
  So $\delta \leqslant \pi^{-1}(\delta)$ (since the union of things less than δ must be $\leqslant \delta$). On the other hand, $\pi^{-1}(\delta) \in \mathcal{M} \preceq L_\delta$. So $\pi^{-1}(\delta) < \delta$. This gives us a contradiction.

- Now to prove that $\beta > 0$. Since $L_\delta \vDash \exists x \forall y \in x (y \neq y)$, we must have $\mathcal{M} \vDash \exists x \forall y \in x(y \neq y)$. Therefore $\mathcal{N}$ believes this too. So $\emptyset \in \mathcal{N} \cap \operatorname{Ord} = \beta$.

- We also prove that β is a limit. Let $L_\delta \vDash \forall \alpha \in \text{Ord}\ \exists x. (x = \alpha + 1)$, then $\mathcal{N} \vDash \forall \alpha \in Ord\ \exists x. (x = \alpha + 1)$. Take $\alpha \in \beta$, then $\alpha \in \mathcal{N}$, so (using absoluteness) $\alpha + 1 \in \mathcal{N} \cap \text{Ord} = \beta$.
- Now, towards proving $\mathcal{N} = L_\beta$, we prove $L_\beta \subseteq \mathcal{N}$. We have $L_\delta \vDash \forall \alpha \in \text{Ord}\ \exists y. (y = L_\alpha)$. Then $\mathcal{N} \vDash \forall \alpha \in \text{Ord}\ \exists y. (y = L_\alpha)$. Since the L_α hierarchy is absolute. $\forall \alpha \in \mathcal{N} \cap \text{Ord} = \beta$, $L_\alpha \in N$.
- Then finally we want to prove $\mathcal{N} \subseteq L_\beta$. We have $L_\delta \vDash \forall x\, \exists y\, \exists z\, (y \in \text{Ord} \wedge z = L_y \wedge x \in z)$. Then $\mathcal{N} \vDash \forall x\, \exists y\, \exists z\, (y \in \text{Ord} \wedge z = L_y \wedge x \in z)$ as well. Fix $a \in \mathcal{N}$, find $\gamma \in \mathcal{N}$ and $z \in \mathcal{N}$ such that $\mathcal{N} \vDash \gamma \in \text{Ord} \wedge z = L_\gamma \wedge a \in z$. Then by absoluteness, $a \in L_\gamma \subseteq L_\beta$.

□

Theorem 12.3. *If $V = L$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α.*

Proof. Assume that $V = L$. We shall show that $\mathcal{P}(\omega_\alpha) \subseteq L_{\omega_{\alpha+1}}$. Then since $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$, the theorem will follow.

To do this, it suffices to show that if $X \subseteq \omega_\alpha$, then there exists $\gamma \in \omega_{\alpha+1}$ such that $X \in L_\gamma$. Take $X \subseteq \omega_\alpha$ and let $\delta > \omega_\alpha$ be a limit ordinal such that $X \in L_\delta$. Now let $\mathcal{M}$ be an elementary submodel of L_δ such that $\omega_\alpha \subseteq \mathcal{M}$, $X \in \mathcal{M}$, $|\mathcal{M}| = \aleph_\alpha$ (using downward Löwenheim-Skolem).

By Gödel's condenstation lemma, take $\mathcal{N}$ to be the Mostowski collapse. There is a limit ordinal $\gamma \leqslant \delta$ such that $\mathcal{N} = L_\gamma$. Since $|\mathcal{N}| = |\mathcal{M}| = \aleph_\alpha$, $|L_\gamma| = \aleph_\alpha$, so $\gamma < \omega_{\alpha+1}$. Now $\omega_\alpha \subseteq \mathcal{M}$, the collapsing map is the identity on ω_α. Then the map fixed X, so $X \in L_\gamma$, and we're done. □

Theorem 12.4 (Gödel)**.**

$$\text{Con}(\mathsf{ZFC}) \Rightarrow \text{Con}(\mathsf{ZFC} + V = L + \mathsf{GCH})$$

Proof. We have shown that there is a definable class L such that

$$\mathsf{ZF} \vdash (\mathsf{ZFC} + V = L + \mathsf{GCH})^L.$$

Suppose $\mathsf{ZFC} + V = L + \mathsf{GCH}$ were inconsistent. Then we can fix φ such that $\mathsf{ZFC} + V = L + \mathsf{GCH} \vdash \varphi \wedge \neg\varphi$. Then

(i) $\mathsf{ZF} \vdash (\mathsf{ZFC} + V = L + \mathsf{GCH})^L$.

(ii) $\vdash (\varphi \wedge \neg\varphi)^{\mathsf{L}}$.

(iii) By relativisation, $\mathsf{ZF} \vdash \varphi^L \wedge (\neg\varphi)^L$.

iv So, $\mathsf{ZF} \vdash \varphi^L \wedge \neg(\varphi^L)$.

(v) Therefore ZF is inconsistent.

Lemma 12.5 (Shepherdson)**.** *There is no class W such that*

$$\mathsf{ZFC} \vdash \text{“}W \text{ is an inner model” and } (\neg\mathsf{CH})^W$$

Proof. Omitted □

Definition 12.6 (Club)**.** Suppose Ω is either a regular cardinal, or Ord, then $C \subset \Omega$ is said to be a **club** if it is:

- Closed: $\forall \gamma \in \Omega \sup(C \cap \gamma) \in C$.
- Unbounded: $\forall \alpha \in \Omega \exists \beta \in C(\beta > \alpha)$.

Definition 12.7 (Stationary)**.** A class $S \subseteq \Omega$ is **stationary** if for every club $C \subseteq \Omega$, $C \cap S \neq \emptyset$.

12.1 $\diamond$ in L

$\diamond$ is the statement that there is a single sequence of length ω_1 which can approximate any subset of ω_1.

Definition 12.8 (Approximate ω_1)**.** We say that a sequence $\langle A_\alpha : \alpha \in \omega_1 \rangle$ **approximates** ω_1, if it is such that

(i) For all $\alpha \in \omega_1$, $A_\alpha \subseteq \alpha$.

(ii) $\forall X \subseteq \omega_1$, $\{\alpha : X \cap \alpha = A_\alpha\}$ is stationary.

Lemma 12.9. $\mathsf{ZF} \vdash \diamond \Rightarrow \mathsf{CH}$.

Proof. Let $\langle A_\alpha \,|\, \alpha \in \omega_1 \rangle$ be a $\diamond$-sequence. Then $\forall X \subseteq \omega$, $\exists \alpha > \omega$. $X = A_\alpha$. Thus $\{A_\alpha \,|\, \alpha \in \omega_1, A_\alpha \subseteq \omega\} = \mathcal{P}(\omega)$. □

Theorem 12.10 (I). *f $V = L$, then $\diamond$ holds.*

Note that $\diamond$ is used in certain inductive constructions to build combinatorial objects (e.g. Suslin trees, more in notes).

12.2 $\square$ in L

Definition 12.11 ($\square_\kappa$). Let κ be an uncountable cardinal, then $\square_\kappa$ is the assertion that there exists a sequence

$$(C_\alpha \mid \alpha \in \lim(\kappa^+))$$

such that

(i) C_α is a club subset of α.

(ii) If $\beta \in \lim(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.

(iii) If $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

Theorem 12.12 (Jensen). *If $V = L$, then $\square_\kappa$ holds ofr every uncountable cardinal κ.*

Proof. This is really really hard, so we won't prove it. $\square$

Lemma 12.13. *If ω_1 holds, then there exists a stationary set*

$$S \subseteq \{\beta \in \omega_2 \mid \mathrm{cf}(\beta) = \omega\}$$

such that for all $\alpha \in \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$, $S \cap \alpha$ is not stationary in α.

Remark. If κ is a weakly compact cardinal, then every stationary subset of κ reflects (i.e. $\exists \alpha \in \kappa$ such that $S \cap \alpha$ is stationary in α). The claim that every stationary subset of $\{\beta \in \omega_2 \mid \mathrm{cf}(\beta) = \omega\}$ reflects at a point of $\mathrm{cf}(\omega_1)$ is equiconsistent with $\mathsf{ZFC} + \exists$ Mahlo cardinal.

Lecture 10

13 Forcing

The idea is to widen our model of ZFC to "add lots of reals." But if we're working over V, then there's nothing to add. Instead, we'll work with countable transitive models (CTMs) of ZFC[1]. If $\mathcal{M}$ is a countable transitive model of ZFC, then we want to add the $\omega_2^{\mathcal{M}}$-many reals to $\mathcal{M}$. We want to do this in a "minimal" way. For example, we don't want to add any new ordinals to $\mathcal{M}$. This gives us much more control over the model that we end up building.

To give some more intuition for what's happening, recall the argument that the sentence $\varphi \equiv \exists x(x^2 = 2)$ is independent of the axioms of fields. This involved starting with a model of $\neg\varphi$, and then extending to $\mathbb{Q}[\sqrt{2}]$ (i.e. extending it in a "minimal" way). The key thing about adding $\sqrt{2}$ is that we also have to add everything that's built from $\sqrt{2}$ and $\mathbb{Q}$ using the axioms of fields.

There are some possible difficulties that could arise:

- Suppose that our countable transitive model (CTM) $\mathcal{M}$, is of the form L_γ where $\gamma \in \omega$ (this isn't actually possible, which is shown in ES 2, but suppose it were for now). Then γ can be coded as a subset of ω, c. Now $c \subseteq \omega$, so it can also be viewed as a real, so if we added c to $\mathcal{M}$, we would also add $\gamma = \mathrm{Ord} \cap \mathcal{M}$.
- Enumerate all formulas as $\{\varphi_n \mid n \in \omega\}$. Let $r = \{n \mid \mathcal{M} \vDash \varphi_n\}$ (I think this is kind of like $0^\#$ but for a CTM). Then we could add a truth predicate to $\mathcal{M}$, so we would have a truth predicate for $\mathcal{M}$ constructible inside $\mathcal{M}$. This is problematic due to Tarski.

The main issues to overcome are the following:

1. We need a method to choose the $\omega_2^{\mathcal{M}}$-many subsets of $\mathcal{M}$.
2. Having chosen them, we need to ensure the extension satisfies ZFC (the hardest part).
3. Why should $\omega_1^{\mathcal{M}}$ and $\omega_2^{\mathcal{M}}$ still be cardinals in the extension?

We'll build these reals *within* our model $\mathcal{M}$. If r is a real, then each of its finite decimal approximations is already in $\mathcal{M}$. The issue is that within $\mathcal{M}$, we do not actually *know* what the real we want to add is. If we could say what it was, then using ZFC, it would already be in $\mathcal{M}$. Instead, we add a "generic" real.

To be generic, we don't want to specify any particular digits (i.e. beginning with 7). It should, however, contain in its decimal expansion any finite sequence (e.g. "746").

[1] Eventually we'll be able to remove this assumption, but first things first.

Definition 13.1 (dense)**.** We call a specification **dense** if any finite approximation can be extended to one satisfying the specification.

So beginning with a 7 is not dense, since if r doesn't begin with a 7, we can't add things to the end to get it to begin with a 7. However, having any finite sequence in your decimal expansion is dense, since you can always extend to add any finite sequence.

It will turn out that being generic corresponds to meeting all dense specifications.

Remark. When forcing, we will use countable transitive models. This means we don't get that $\text{Con}(\mathsf{ZFC}) \Rightarrow \text{Con}(\mathsf{ZFC}+\neg\mathsf{CH})$, because we've added countable transitive models, but we will then use the reflection theorem to obtain the result we want.

Remark. The axiom of choice is not needed in the basic machinery, so we will primarily work over ZF, and state where choice is used.

13.1 Partial Orders

The chapter without which every set theory and mathematical logic course is incomplete.

Definition 13.2 (Pre-order)**.** A **preorder** is a pair

$$(\mathbb{P}, \leqslant)$$

such that

- $\mathbb{P}$ is nonempty.
- $\leqslant$ is a binary relation.
- $\leqslant$ is transitive ($p \leqslant q \wedge q \leqslant r \Rightarrow p \leqslant r$)
- $\leqslant$ is reflexive ($p \leqslant p$).

A preorder is a **partial order** if $\leqslant$ is additionallty anti-symmetric ($p \leqslant q \wedge q \leqslant p \Rightarrow p = q$).

Definition 13.3 (Forcing poset)**.** A **forcing poset** is a triple $(\mathbb{P}, \leqslant_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, where

- $(\mathbb{P}, \leqslant_{\mathbb{P}})$ is a pre-order

- $\mathbb{1}_{\mathbb{P}}$ is a maximal element.

Elements of $\mathbb{P}$ are called **conditions**.

- q is **stronger** than p, if $q \leqslant_{\mathbb{P}} p$ also written that q is an **extension** of p.
- p and q are **compatible**, written $p||_{\mathbb{P}}q$ if there exists r such that $r \leqslant_{\mathbb{P}} p, q$. Otherwise they are called **incompatible**, written $p \perp_{\mathbb{P}} q$.

In some texts forcing is written the other way round. This is called the Jerusalem notation.

Also note that $\mathbb{P} \in \mathcal{M}$ abbreviates $(\mathbb{P}, \leqslant_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}) \in \mathcal{M}$. (transitivity would give that $\mathbb{1}_{\mathbb{P}} \in \mathbb{P}$, but not necessarily $\leqslant_{\mathbb{P}}$, but whenever we talk about it in this course, it's fine).

Definition 13.4 (Separative)**.** A pre-order is **separative** iff for all distinct p, q exactly one of the following two conditions hold:

- Either $q \leqslant p$ and $p \not\leqslant q$.
- (X)or $q > p$ and there is an $r \leqslant q$ such that $r \perp p$

Proposition 13.5. *If $(\mathbb{P}, \leqslant)$ is a separative pre-order, then $(\mathbb{P}, \leqslant)$ is a partial order.*

Proof. Third (probably?) Example sheet.

Proposition 13.6. *Suppose that $(\mathbb{P}, \leqslant)$ is a pre-order. Define $p \sim q$ by*

$$p \sim q \Leftrightarrow \forall r \in \mathbb{P}\,(r||p \leftrightarrow r||q).$$

Then there is a separative pre-order on $\mathbb{P}\backslash \sim$ such that

- *$[p] \perp [q]$ iff $p \perp q$*
- *If $\mathbb{P}$ has a maximal element so does $\mathbb{P}/\sim$.*

Example 13.7. For sets I and J, let $\mathrm{Fn}(I, J)$ denote the set of all finite partial functions from I to J. That is

$$\mathrm{Fn}(I, J) = \{p : |p| < \omega,\ p \text{ is a function},\ \mathrm{dom}(P) \subseteq I,\ \mathrm{ran}(P) \subseteq J\}$$

Let $\leqslant$ be the revesrse inclusion of $\mathrm{Fn}(I,J)$, so $q \leqslant p$ iff $q \supseteq p$. Also $\mathbb{1}_{\mathbb{P}} = \emptyset$. Then $(\mathrm{Fn}(I,J), \leqslant, \emptyset)$ is a forcing poset, and moreover the pre-order is separative.

Lecture 11 Note that when α is an ordinal, $\mathrm{Fn}(\alpha \times \omega, 2)$ is often written $\mathrm{Add}(\omega, \alpha)$.

Remark. Note on reals: Let $\mathbb{R}$ be your favourite construction of the reals. There are explicit absolute bijections

$$f : \mathcal{P}(\omega) \to {}^{\omega}\omega$$
$$g : {}^{\omega}\omega \to {}^{\omega}2$$
$$h : {}^{\omega}2 \to \mathbb{R}$$

So, in $\mathcal{M} \vDash \mathsf{ZFC}$, knowledge of $\mathcal{P}^{\mathcal{M}}(\omega)$ gives knowledge of $({}^{\omega}\omega)^{\mathcal{M}}$, $({}^{\omega}2)^{\mathcal{M}}$, $\mathbb{R}^{\mathcal{M}}$. So, in order to add a real to $\mathcal{M}$ it suffices to add a function in ${}^{\omega}\omega$, or a subset of ω. We can freely switch between these sets. In formal arguments, reals will normally be either subsets of ω or functions $\omega \to 2$.

Back to the forcing poset, $\mathrm{Fn}(I,J)$ (the collection of partial functions from I to J).

Definition 13.8 (Chains and Antichains)**.** Let $\mathbb{P}$ be a forcing poset

- A **chain** is a subset $C \subseteq \mathbb{P}$ such that $\forall p, q \in C (p \leqslant q \vee q \leqslant p)$.
- A **antichain** is a subset $A \subseteq \mathbb{P}$ such that $\forall p, q \in A.\ (p \perp q)$.
- An antichain is **maximal** if it is not a proper subset of any other antichain of $\mathbb{P}$.
- $\mathbb{P}$ has the countable chain condition, (CCC), if every antichain in $\mathbb{P}$ is countable.

Example. The set of functions $\{\{(0,0),(1,n)\} \mid n \in \omega\}$ forms an antichain of length ω in $\mathrm{Fn}(2,\omega)$.

Definition 13.9 (Delta system/Root of a delta system)**.** A family of sets $\mathcal{A}$ forms a **Delta system with root** R iff $X \cap Y = R$ for all $X \neq Y$ in $\mathcal{A}$.

Definition 13.10 (Subsets of a set of cardinality θ)**.** Let A be a set θ a cardinal. Then $[A]^{\theta}$ is the set of subsets of A of size θ:

$$[A]^{\theta} = \{x \subseteq A \mid |x| = \theta\}$$

. Naturally, $[A]^{<\theta}$ is defined as:

$$[A]^{<\theta} = \{x \subseteq A \mid |x| < \theta\}$$

Recall the theorem that if κ is a regular cardinal, and $\mathcal{F}$ is a family of sets with $|\mathcal{F}| < \kappa$, then $|X| < \kappa$ for all $X \in \mathcal{F}$, then $|\cup \mathcal{F}| < \kappa$.

Lemma 13.11 (Delta System Lemma, ZFC)**.** *Let κ be an uncountable regular cardinal, let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$, then there exists $\mathcal{B} \in [A]^\kappa$ such that B forms a delta system.*

Proof. To begin with, construct $C \in [A]^\kappa$ such that all elements of C have the same size. By assumption $|X| < \aleph_0$ for all $X \in \mathcal{A}$. So set $Y_n = \{X \in \mathcal{A} \mid |X| = n\}$. If $|Y_n| < \kappa$ for each $n < \kappa$, then $|\mathcal{A}| = |\cup Y_n| < \kappa$. This is a contradiction, so some Y_n must have size κ.

Now fix $n \in \omega$ such that $C = Y_n$ has size κ. We proceed inductively on n to prove that if $C = \{X : |X| = n\}$ has size κ, then there is a $\mathcal{B} \subseteq \mathcal{C}$ of size κ such that $\mathcal{B}$ forms a Delta system.

Firstly, if $n = 1$, then C must be a family of pairwise disjoint singletons, so forms a Delta system. Now assume $n > 1$, the claim holds for $n - 1$. For each $p \in \cup C$ let $C_p = \{X \in C \mid p \in X\}$. Tjere are twp cases:

- Suppose $|C_p| = \kappa$ for some $p \in \cup C$. Fix such a p, and set $\mathcal{D} = \{X \setminus \{p\} \mid X \in C_p\}$. Then $\mathcal{D}$ has size κ and each element of $\mathcal{D}$ has size $n - 1$. So by the inductibe hypothesis, w can find $\mathcal{E} \in [\mathcal{D}]^\kappa$ such that $\mathcal{E}$ forms a Delta system with root R. Then $\{Y \cup \{p\} \mid Y \in \mathcal{E}\} \in [C]^\kappa$ forms a delta system with root $R \cup \{p\}$.
- Suppose $|C_p| < \kappa$ for all $p \in \cup C$. Since κ is regular, for any set S with $|S| < \kappa$, $\{X \in C \mid X \cap S \neq \emptyset\} = \bigcup_{p \in S} C_p$ has size less than κ.

  Then there must exist $X \in C$ such that $X \cap S = \emptyset$. We recursively choose $X_\alpha \in C$ for $\alpha < \kappa$ such that

  $$X_\alpha \cap \bigcup_{\beta < \alpha} X_\beta = \emptyset.$$

  This means that $\{X_\alpha \mid \alpha \in \kappa\} \in [C]^\kappa$ is a Delta system with empty root.

□

Proposition 13.12. *If κ were either ω or a singular cardinal, then we could find a family of finite sets with $|\mathcal{A}| = \kappa$ and no $\mathcal{B} \in [\mathcal{A}]^\kappa$ forms a delta system.*

Proof. Exercise. $\square$

Lemma 13.13 (ZFC)**.** $\mathrm{Fn}(I, J)$ *has the countable chain condition iff $I = \emptyset$ or J is countable.*

Proof. Observe that if $I = \emptyset$, or $J = \emptyset$, then $\mathrm{Fn}(I, J) = \emptyset$, so trivially has the CCC.

Now assume that $I, J \neq \emptyset$.

($\Rightarrow$) Suppose J were uncountable. Then fix some $i \in I$, and look at $\{\{(i, j)\} \mid j \in J\}$, this forms an uncountable antichain.

($\Leftarrow$) Suppose J was countable, and let $\{p_\alpha \mid \alpha \in \omega_1\}$ be a collection of distinct elements of $\mathrm{Fn}(I, J)$. Let $\mathcal{A} = \{\mathrm{dom}(p_\alpha) \mid \alpha \in \omega_1\}$. Then by the Delta System Lemma, there exists an uncountable set $\mathcal{B} \in [\mathcal{A}]^{\omega_1}$ with root $R \subseteq I$.

Since $R \subseteq \mathrm{dom}(p_\alpha)$ for all $\mathrm{dom}(p_\alpha) \in \mathcal{B}$, R is finite. Since J is countable, there are only countably many functions from R to J. So, since $\mathcal{B}$ is uncountable, we can find $\alpha \neq \beta$ such that $\mathrm{dom}(p_\alpha)$ and $\mathrm{dom}(p_\beta)$ are both in $\mathcal{B}$, and they agree with each other on R (i.e. $p_\beta \restriction R = p_\alpha \restriction R$). But then, since R was a root of this system, this means that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta)$. Therefore, $p_\alpha || p_\beta$ (since $p_\alpha \cup p_\beta = R$).

$\square$

13.2 Dense sets and Genericity

Definition 13.14 (Dense/Open/Filter)**.** Let $\mathbb{P}$ be a forcing poset. Then

- $D \subseteq \mathbb{P}$ is **dense** iff $\forall p \in \mathbb{P}\, \exists q \in D\ (q \leqslant p)$.
- $D \subseteq \mathbb{P}$ is **open** iff $\forall p \in D\, \forall q \in \mathbb{P}\, (q \leqslant p \to q \in D)$.
- $\mathcal{G} \subseteq \mathbb{P}$ is a **filter** on $\mathbb{P}$ iff
  - (i) $\mathbb{1}_{\mathbb{P}} \in \mathcal{G}$ (non-empty)
  - (ii) $\forall p, q \in \mathcal{G}\ \exists r \in \mathcal{G}\ (r \leqslant p \wedge r \leqslant q)$ (directed)
  - (iii) $\forall p, q \in \mathbb{P}\ (q \leqslant p \wedge q \in \mathcal{G} \to p \in \mathcal{G})$ (upwards closed)

Definition 13.15 ($\mathbb{P}$-generic over a model)**.** Let $\mathbb{P}$ be a forcing poset. $\mathcal{G}$ is said to be **$\mathbb{P}$-generic over** $\mathcal{M}$ if G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for every denset $D \subset \mathbb{P}$ such that $D \in \mathcal{M}$.

Lemma 13.16 (Generic Filter existence)**.** *Let $\mathcal{M}$ be an elementary countable set, and $\mathbb{P} \in \mathcal{M}$ be a forcing poset. Then for all $p \in \mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ with $p \in G$ such that $\mathcal{G}$ is $\mathbb{P}$-generic over $\mathcal{M}$.*

Proof. In our external universe, let $(D_n \mid n \in \omega)$ enumerate all dense subsets of $\mathbb{P}$ which lie in $\mathcal{M}$. We inductively define $X \subseteq \mathbb{P}$, $X = \{q_n \mid n \in \omega\}$ as follows:

- $q_0 = p$
- Given q_n, choose $q_{n+1} \in D_n$ such that $q_{n+1} \leqslant q_n$. Let $G = \{r \in \mathbb{P} \mid \exists n. (q_n \leqslant r)\}$.

Then it is clear to see that G is a filter:

- Clear to see that $p \leqslant \mathbb{1}$, so $\mathbb{1} \in G$.
- If $r, r' \in G$, then $q_n \leqslant r$, $q_m \leqslant r'$. Then $q_{\max(m,n)} \leqslant r, r'$.
- If $s \leqslant r$ and $s \in G$, then $q_n \leqslant s \leqslant r$ gives $r \in G$. Then G is a generic filter.

Lecture 12

Definition 13.17 (Minimal element)**.** A $p \in \mathbb{P}$ is **minimal** if $\forall q \in \mathbb{P}. \neg(q < p)$.

Lemma 13.18. *Let $\mathcal{M}$ be a countable model of* ZF *and $\mathbb{P} \in \mathcal{M}$ is a separative partial order. Then either $\mathbb{P}$ has a minimal element, or for every $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over $\mathcal{M}$, $G \notin \mathcal{M}$.*

Proof. Suppose $\mathbb{P}$ does not have a minimal element, and that $G \subseteq \mathbb{P}$ is generic. Let $\mathcal{F} \in \mathcal{M}$ be a filter, $F \subseteq \mathbb{P}$, then $D_{\mathcal{F}} := \mathbb{P} \setminus \mathcal{F}$ is in $\mathcal{M}$, and is a dense set.

Then $G \cap D_{\mathcal{F}} \neq \emptyset$ fora ll filters $\mathcal{F} \in \mathcal{M}$. So $G \neq \mathcal{F}$ and thus $G \neq \mathcal{M}$. Fix $p \in \mathbb{P}$. If $p \notin \mathcal{F}$, then $p \in D_{\mathcal{F}}$ Otherwise, suppose $p \in \mathcal{F}$, since p is not minimal, fix $q < p$. Then $p \nleq q$, so since $\mathbb{P}$ is separative, there is $r \leqslant p$ such that $r \perp q$. But all conditions in $\mathcal{F}$ are compatible (WHY????), so one of r and q is not in $\mathcal{F}$. ∎

Proposition 13.19. *For sets I, J with $|I| \geqslant \omega$ and $|J| \geqslant 2$.* $\mathrm{Fn}(I, J)$ *is a separative partial order without a minimal element.*

Proposition 13.20 (ZFC)**.** *Suppose $\mathbb{P} \in \mathcal{M}$ is a forcing poset, and $G \subseteq \mathbb{P}$, then TFAE:*

(i) $\forall D \in \mathcal{M}. (D$ *dense in* $\mathbb{P} \to G \cap D \neq \emptyset)$.

(ii) $\forall p \in G \, \forall D \in \mathcal{M}. (D$ *is dense below* $p \cap bbP \to G \cap D \neq \emptyset)$

(iii) $\forall D \in \mathcal{M}. (D$ *is open dense in* $\mathbb{P} \to G \cap D \neq \emptyset)$.

(iv) $\forall D \in \mathcal{M}. (D$ *is a maximal antichain in* $\mathbb{P} \to G \cap D \neq \emptyset)$.

Proof. Example Sheet 3 lol. ☐

14 Names

Definition 14.1 ($\mathbb{P}$-names)**.** Let $\mathbb{P}$ be a forcing poset, we define the class of **$\mathbb{P}$-names**, $\mathcal{M}^{\mathbb{P}}$ recursively as follows:

(i) $\mathcal{M}_0^{\mathbb{P}} = \emptyset$.

(ii) $\mathcal{M}_{\alpha+1}^{\mathbb{P}} = \mathcal{P}^{\mathcal{M}}(\mathbb{P} \times \mathcal{M}_\alpha^{\mathbb{P}})$

(iii) $\mathcal{M}_\lambda^{\mathbb{P}} = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha^{\mathbb{P}}$

(iv) $\mathcal{M}^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{Ord}} M_\alpha^{\mathbb{P}}$.

An element of $\mathcal{M}^{\mathbb{P}}$ is called a **$\mathbb{P}$-name**, denoted by $\dot{x}$. Given $x \in \mathcal{M}^{\mathbb{P}}$, $\mathrm{ran}(\dot{x}) = \{\dot{y} \mid \exists p \in \mathbb{P}, \langle p, \dot{y} \rangle\}$.

Remark. Alternatively, by transfinite recursion on rank, the class of $\mathbb{P}$-names over V, $V^{\mathbb{P}}$ is defined in the following way:

If $\mathrm{rank}(\dot{x}) = \alpha$, then $x \in V^{\mathbb{P}}$ iff $\dot{x}$ is a relation and $\forall \langle p, \dot{y} \rangle \in \dot{x}$ $(p \in \mathbb{P}, y \in V^{\mathbb{P}} \cap V_\alpha)$. Then $\mathcal{M}^{\mathbb{P}} = V^{\mathbb{P}} \cap \mathcal{M}$.

Definition 14.2 ($\mathbb{P}$-rank)**.** The **$\mathbb{P}$-rank**, $\mathrm{rank}_{\mathbb{P}}(\dot{x})$ is the least α such that $\dot{x} \subseteq \mathbb{P} \times \mathcal{M}_\alpha$.

Definition 14.3 (Interpretation of $\dot{x}$ by G)**.** Let $\dot{x}$ be a $\mathbb{P}$-name, and $G \subseteq \mathbb{P}$. We define the **interpretation of $\dot{x}$ by G** recursively as:

$$\dot{x}^G = \{\dot{y}^G \mid \exists p \in G. (\langle p, \dot{y}\rangle \in \dot{x})\}$$

Definition 14.4 (Forcing extension)**.** The **forcing extension** of $\mathcal{M}$ by G, $\mathcal{M}[G]$ is

$$M[G] = \{\dot{x}^G \mid \dot{x} \in \mathcal{M}^{\mathbb{P}}\}.$$

Example.

- $\emptyset \in \mathcal{M} \Rightarrow \emptyset^G = \emptyset$.
- $\dot{x} = \{\langle p, \emptyset\rangle, \langle r, \{\langle q, \emptyset\rangle\}\rangle\}$. If all of $p, q, r \in G$, then

$$\begin{aligned}\dot{x}^G &= \{(\langle p, \emptyset\rangle)^G, (\langle r, \{\langle q, \emptyset\rangle\}\rangle)^G\} \\ &= \{\emptyset, \{(\langle q, \emptyset\rangle)\}^G\} \\ &= \{\emptyset, \{\emptyset\}\}\end{aligned}$$

  On the other hand, if $p, r \in G$, then $\dot{x}^G = \emptyset$.
  If $r \in G, p, q \notin G$, then $\dot{x}^G = \{(\langle q, \emptyset\rangle)^G\} = \{\emptyset\}$.
  And finally, if $p \in G$ and $r \notin G$, then $\dot{x}^G = \{\emptyset\}$.

15 Generic Model Theorem

Theorem 15.1 (Generic Model Theorem)**.** *Let $\mathcal{M}$ be a countable transitive model of* ZF*. Let $\mathbb{P}$ be a forcing poset, and G be a $\mathbb{P}$-generic filter. Then*

(i) $\mathcal{M}[G]$ *is a transitive set.*

(ii) $|\mathcal{M}[G]| = \aleph_0$.

(iii) $\mathcal{M}[G] \vDash \mathsf{ZFC}$*, and if* $\mathcal{M} \vDash \mathsf{AC}$ *then* $M[G] \vDash \mathsf{AC}$.

(iv) $\mathrm{Ord}^{\mathcal{M}} = \mathrm{Ord}^{\mathcal{M}[G]}$

(v) $\mathcal{M} \subseteq \mathcal{M}[G]$

(vi) $\mathcal{M}[G]$ *is the smallest countable transitive model of* ZF *such that* $\mathcal{M} \subseteq \mathcal{M}[G]$ *and* $G \in \mathcal{M}[G]$.

This is a big theorem, that we will prove over multiple lectures, and in multipl small parts.

15.1 Canonical names

Definition 15.2 (Canonical name)**.** Given $(\mathbb{P}, \leqslant, \mathbb{1})$ ad set $x \in \mathcal{M}$, we recursively define the **canonical name** of x, $\check{x}$ as

$$\check{x} = \{\langle \mathbb{1}, \check{y}\rangle \mid y \in x\}.$$

(pronounced x-check).

Lemma 15.3. *If $\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$ and $\mathbb{1} \in G \subseteq \mathbb{P}$¿ Then*

(i) $\forall x \in \mathcal{M}(\check{x} \in \mathcal{M}^{\mathbb{P}}$ *and* $\check{x}^G = x)$.

(ii) $\mathcal{M} \subseteq M[G]$.

(iii) $\mathcal{M}[G]$ *is transitive.*

Proof. (ii) follows from (i). (i) is proven inductively. We prove that $\check{x} \in \mathcal{M}^{\mathbb{P}}$ by using the definition of $\mathbb{P}$-names by transfinite recursion. Next $\check{x}^G = \{\check{y}^G \mid y \in x\} = x$.

To prove that $\mathcal{M}[G]$ is transitive, suppose that $x \in y$ and $y \in \mathcal{M}[G]$. Then by definition, $y = \dot{y}^G$ for some $\dot{y} \in \mathcal{M}^{\mathbb{P}}$. By construction, any element of y is of the form $\dot{z}^G$. So in particular, we must have that $x = \dot{x}^G$ for some $\dot{x} \in \mathcal{M}^{\mathbb{P}}$. So $\dot{x}^G \in \mathcal{M}[G]$. ☐

Remark. Even if $G \notin \mathcal{M}$, we can still define a name for G in $\mathcal{M}$. From this, it follows that if $G \notin \mathcal{M}$, then $\mathcal{M}[G] \neq \mathcal{M}$.

Proposition 15.4 (L)**.** *et* $\dot{G} = \{\langle p, \check{p}\rangle \mid p \in \mathbb{P}\}$*. Then* $G = \dot{G}^{\mathbb{P}} \in M[G]$.

Proof.

$$\begin{aligned}\dot{G}^G &= \{\check{p}^G \mid p \in G\} \\ &= \{p \mid p \in G\} \\ {}^{=}G\end{aligned}$$

☐

Definition 15.5 (Unordered pairs/ordered pairs)**.** Given $\dot{x}, \dot{y} \in \mathcal{M}^{\mathbb{P}}$, let $\mathrm{up}(\dot{x}, \dot{y}) = \{\langle \mathbb{1}, \dot{x}\rangle, \langle \mathbb{1}, \dot{y}\rangle\}$, and let $\mathrm{op}(\dot{x}, \dot{y}) = \mathrm{up}(\mathrm{up}(\dot{x}, \dot{x}), \mathrm{up}\, \dot{x}, \dot{y})$. Very similar to how we define kuratowski pairs.

Proposition 15.6. *For any $\dot{x}, \dot{y} \in \mathcal{M}^{\mathbb{P}}$, and $G \subseteq \mathbb{P}$, $\mathbb{1} \in G$:*

$$(\mathrm{up}(\dot{x}, \dot{y}))^G = \{\dot{x}^G, \dot{y}^G\}$$
$$(\mathrm{op}(\dot{x}, \dot{y}))^G = \langle \dot{x}^G, \dot{y}^G\rangle$$

Proof. Omitted □

15.2 First results towards showing Generic Model Theorem

Lemma 15.7. *Suppose $\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$ is a forcing poset. Then if $G \subseteq \mathbb{P}$ and $\mathbb{1} \in G$, then $\mathcal{M}[G]$ is a transitive model of*

- *Extensionality*
- *Empty set*
- *Foundation*
- *Pairing*

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Lemma 15.8 (S)**.** *uppose $\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$ is a forcing poset. Then if $G \subseteq \mathbb{P}$ and $\mathbb{1} \in G$, we have*

(i) $\mathrm{rank}(x^G) \leqslant \mathrm{rank}(x)$ *for all $x \in \mathcal{M}^{\mathbb{P}}$.*

(ii) $\mathrm{Ord}^{\mathcal{M}} = \mathrm{Ord}^{\mathcal{M}[G]}$.

(iii) $|\mathcal{M}[G]| = |\mathcal{M}|$.

Proof.

(i) We prove this by induction:

- $\emptyset^G = \emptyset$ and $\mathrm{rank}(\emptyset^G) = \mathrm{rank}(\emptyset) = 0$.

- For the inductive step, we have:

$$\begin{aligned}\mathrm{rank}(\dot{x}^G) &= \sup\{\mathrm{rank}(u)+1 \mid u \in \dot{x}^G\} \\ &\leqslant \sup\{\mathrm{rank}(\dot{y}^G)+1 \mid \dot{y} \in \mathrm{ran}(\dot{x})\} \\ &\leqslant \sup\{\mathrm{rank}(\dot{y})+1 \mid \dot{y} \in \mathrm{ran}(\dot{x})\} && \text{(IH)} \\ &\leqslant \sup\{\mathrm{rank}(u)+1 \mid u \in \dot{x}\} \\ &\leqslant \mathrm{rank}(\dot{x}).\end{aligned}$$

(ii) Since $\mathcal{M} \subseteq \mathcal{M}[G]$, and also being an ordinal is an absolute property, $\mathrm{Ord}^{\mathcal{M}} \subseteq \mathrm{Ord}^{\mathcal{M}[G]}$. For the reverse inclusion, take $\alpha \in \mathrm{Ord}^{\mathcal{M}[G]}$ and fix $\dot{x} \in \mathcal{M}^{\mathbb{P}}$ such that $\alpha = \dot{x}^G$. Then $\alpha = \mathrm{rank}(\alpha) \leqslant \mathrm{rank}(\dot{x})$. So, since $\mathcal{M}$ is transitive, $\alpha \in \mathrm{Ord}^{\mathcal{M}}$.

(iii) Since any element of $\mathcal{M}[G]$ is of the form $\dot{x}^G$ for some $\dot{x} \in \mathcal{M}^{\mathbb{P}} \subseteq \mathcal{M} \subseteq \mathcal{M}[G]$. Therefore,

$$|\mathcal{M}[G]| \leqslant |\mathcal{M}^{\mathbb{P}}| \leqslant |\mathcal{M}| \leqslant |\mathcal{M}[G]|.$$

This follows, since $\dot{x} = (\check{\dot{x}})^G$

□

Corollary 15.9. *$M[G]$ models the axiom of infinity.*

Proof. Omitted □

Lemma 15.10 (S)**.** *uppose $\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$, $G \subseteq \mathbb{P}$, $\mathbb{1} \in G$. Then if $\mathcal{N}$ is any other transitive model of* ZF *containing $\mathcal{M}$ as a definable class in $\mathcal{N}$, and $G \in \mathcal{N}$, then $M[G] \subseteq \mathcal{N}$.*

Proof. If $\mathcal{N} \supseteq \mathcal{M}$ is a transitive model of ZF with $G \in \mathcal{N}$, we carry out the construction of $\mathcal{M}[G]$ in $\mathcal{N}$. Namely, show that for all $\dot{x} \in \mathcal{M}^{\mathbb{P}}$, $\dot{x}^G \in \mathcal{N}$ by induction on ranks.

- First, sicne the empty set axiom holds in $\mathcal{N}$, $\emptyset^G = \emptyset \in \mathcal{N}$.

  Moreover, since $\mathcal{M}^{\mathbb{P}} = V^{\mathbb{P}} \cap \mathcal{M} \subseteq \mathcal{V}^{\mathbb{P}} \cap \mathcal{N} = \mathcal{N}^{\mathbb{P}}$. So if $\dot{x} \in \mathcal{M}^{\mathbb{P}}$, then $\dot{x} \in \mathcal{N}^{\mathbb{P}}$, and in particular $\dot{x} \in \mathcal{N}$.

- For the inductive step, suppose that for every $\langle p, \dot{y}\rangle \in \dot{x}$, $\dot{y}^G \in \mathcal{N}$:

$$\begin{aligned}\left((\dot{x})^G\right)^{\mathcal{N}} &= \left(\{\dot{y}^G \mid \exists p \in G.\, (\langle p, \dot{y}\rangle \in \dot{x})\}\right)^{\mathcal{N}} \\ &= \left(\{(\dot{y}^g)^{\mathcal{N}} \mid (\exists p \in G\langle p, \dot{y}\rangle \in \dot{x})\}\right)^{\mathcal{N}} \\ &= \{\dot{y}^G \mid \exists p \in G.\, \langle p, \dot{y}\rangle \in \dot{x}\} \\ &= \dot{x}^G\end{aligned}$$

□

Lemma 15.11. *Suppose $\mathcal{M}$, $\mathbb{P}$, G as above. Additionally, assume that G is a filter. Then $\mathcal{M}[G]$ satisfies unions.*

Proof. It suffices to prove that for all $a \in \mathcal{M}[G]$, there is some $b \in \mathcal{M}[G]$ such that $\cup a = b$. In order to show this, fix $\dot{a} \in \mathcal{M}^{\mathbb{P}}$ such that $\dot{a}^G = a$ and let $\dot{b} = \{\langle p, \dot{z}\rangle \mid \exists\langle q, \dot{y}\rangle \in \dot{a}, \exists r \in \mathbb{P}(\langle r, \dot{z}\rangle \in \dot{y} \wedge p \leqslant r, q)\}$.

Observe that $\dot{b} \in \mathcal{M}^{\mathbb{P}}$. Since $\dot{a}$ is a $\mathbb{P}$ name, any $adoty \in \operatorname{ran}(a)$ is a $\mathbb{P}$-name. Then $\dot{b}$ consists of pairs $\langle p, \dot{x}\rangle$ where $p \in \mathbb{P}$, $\dot{z} \in \operatorname{ran} \dot{y}$ for some $y \in \operatorname{ran}(\dot{a})$ Thus $\dot{z} \in V^{\mathbb{P}}$. Moreover, $b \in \mathcal{M}$ since $dotb \subseteq \mathbb{P} \times \operatorname{tcl}(\dot{a})$.

Now we need to show that $\cup a \subseteq \dot{b}^G$. Take $w \in \cup a$. Then $w \in v$ for some $v \in a$. Since $\mathcal{M}[G]$ is transitive, $w, v \in \mathcal{M}[G]$. So for $\dot{y}, \dot{z} \in \mathcal{M}^{\mathbb{P}}$. and conditions $q, r \in G$ such that $w = \dot{z}^G$, $v = \dot{y}^G$, $\langle q, \dot{y}\rangle \in \dot{a}$, $\langle r, \dot{x}\rangle \in \dot{y}$.

Now, since G is a filter, by downwards directedness, fix $p \leqslant q, r$, $p \in G$. But then $\langle p, \dot{x}\rangle \in \dot{b}$ and $w = \dot{z}^G \in \dot{b}^G$.

Now we want to show that $\dot{b}^G \subseteq \cup a$. Take $c \in \dot{b}^G$ and fix $\langle p, \dot{x}\rangle \in \dot{b}$ such that $p \in G$ and $\dot{z}^G = c$.

By definition, fix $\langle q, \dot{y}\rangle \in \dot{a}$ and $r \in \mathbb{P}$ such that $\langle r, \dot{z}\rangle \in \dot{y}$, and $p \leqslant q, r$. Since G is a filter, it is upwards closed, so $q, r \in G$. Therefore $\dot{z}^G \in \dot{y}^G$, $\dot{y}^G \in \dot{a}^G$. So $c \in \dot{y}^G$ for some $\dot{y}^G \in a$. □

15.3 Moivation for Genericity of Generic Model Theorem

Suppose $\mathbb{P} \in \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{M}[G]$. Then $\mathbb{P}, G \in \mathcal{M}[G]$. If $\mathcal{M}[G]$ models anything reasonable, $\mathbb{P} \setminus G \in \mathcal{M}[G]$. So if we try to build a name for $\mathbb{P} \setminus G$, a natural name is $c = \{\langle q, \check{p}\rangle \mid p, q \in \mathbb{P}, p \perp q\}$. Then $\dot{c}^G = \{p \mid \exists q \in G, p \perp q\}$. Now if G is a filter, its elements are pairwise compatible, so $G \cap \dot{c}^G = \emptyset$. But we still need $G \cup \dot{c}^G = \mathbb{P}$.

Now, for each $p \in \mathbb{P}$, set $D_p = \{q \in \mathbb{P} \,|\, p \perp q \vee q \leqslant p\}$. It is easy to check that $D_p \in \mathcal{M}$ is dense. Now if G were generic, we could fix $q \in G \cap D_p$. Then if $p \perp q$, $p \in \dot{c}^G$, and if $q \leqslant p$, then $p \in G$. Thus $G \cup \dot{c}^G = \mathbb{P}$.

Proposition 15.12. *Suppose $\mathcal{M}$ is a countable transitive model of* ZF*. Then there exists a $\mathbb{P} \in \mathcal{M}$ and a (non-generic) filter $G \subseteq \mathbb{P}$ such that $\mathbb{P} \setminus G \notin \mathcal{M}[G]$.*

Proof. Discussed in example sheet 3. □

15.4 Forcing relation

To show separation, suppose $\varphi(x, y)$ were a formula, $a, b \in \mathcal{M}^{\mathbb{P}}$, need to show that $C = \{z \in \dot{a}^G \,|\, (\varphi(z, \dot{b}^G))^{\mathcal{M}[G]}\} \in \mathcal{M}[G]$. This is unclear even for $\varphi(x, y) \equiv x \notin y$. We will build a way to reason about when φ holds from within $\mathcal{M}$ without having to rely on G.

To do this, define a relation between conditions and names in $V^{\mathbb{P}}$, written $p \Vdash \varphi$. Its relativisation $(p \Vdash \varphi)^{\mathcal{M}}$ will give us a way to work in $\mathcal{M}$.

Our aim is to define $\Vdash$ such that: $p \Vdash \varphi(\dot{u}) \Leftrightarrow \forall G \subseteq \mathbb{P}$ with G generic, and $p \in G$, $\mathcal{M}[G] \vDash \varphi(\dot{u}^G)$. A naive definition might be to define $\langle p, \dot{x}\rangle \in \dot{y} \Rightarrow p \Vdash \dot{x} \in \dot{y}$. Why not $\Leftarrow$. Consider $\dot{x} = \{\langle p, \varphi\rangle\}$ where $p \neq \mathbb{1}$, Then $p \Vdash \emptyset \in \dot{x}$. Suppose $q \perp p$, then we will have $q \Vdash \dot{x} = \emptyset$, and $q \Vdash \dot{x} \in \check{1}$. But $\langle q, \dot{x}\rangle \notin \check{1} = \{\langle \mathbb{1}, \emptyset\rangle\}$. (Note that generics meet dense sets, so suffices to consider dense sets).

Definition 15.13 ($\mathbb{P}$-forcing language)**.** For a forcing poset $\mathbb{P}$, the **$\mathbb{P}$-forcing language** $\mathcal{FL}_{\mathbb{P}}$ is the class of logical formulas using the binary relation $\in$, and constant symbols from $V^{\mathbb{P}}$.

Recall that D is dense below p iff $\forall q \leqslant p \, \exists s \in D.\, s \leqslant q$. T

Definition 15.14 (Forcing relation)**.** Let $p \in \mathbb{P}$, and $\dot{x}, \dot{y}, \dot{u} \in V^{\mathbb{P}}$, we define the **forcing relation**, $p \Vdash \varphi(\dot{u})$ recursively as follows:

- $p \Vdash \varphi(\dot{u}) \wedge \psi(\dot{u})$ iff $p \Vdash \varphi(\dot{u})$ and $p \Vdash \psi(\dot{u})$.
- $p \Vdash \neg\varphi(\dot{u})$ iff tyhere is no $q \leqslant p$ such that $q \Vdash \varphi(\dot{u})$.
- $p \Vdash \exists x \varphi(x, \dot{u})$ iff the set $\{q \leqslant p \,|\, \exists \dot{x} \in V^{\mathbb{P}}, q \Vdash \varphi(\dot{x}, \dot{u})\}$ is dense below p.
- $p \Vdash \dot{x} \subseteq \dot{y}$ iff for all $\langle q_1, \dot{z}_1\rangle \in \dot{x}$, $\{r \leqslant p \,|\, r \leqslant q_1 \rightarrow \exists \langle q_2, \dot{z}_2\rangle \in \dot{y}.\, r \leqslant q_2 \wedge r \Vdash \dot{z}_1 = \dot{z}_2\}$ This is what was written but probably still check notes online for this section.

The idea of this inclusion definition is that for everything in the range of $\dot{x}$, there is something in the range of $\dot{y}$, and some $r \in G$ such that $r \Vdash \dot{z}_1 = \dot{z}_2$, and $\langle q_1, \dot{z}_1 \rangle \in \dot{x}$, $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$. So we want the r to satisfy $r \leqslant p, q_1, q_2$.

- $p \Vdash \dot{x} \in \dot{y}$ iff $\{q \leqslant p \mid \exists \langle r, \dot{z} \rangle \in \dot{y}(q \leqslant r \wedge q \Vdash \dot{x} = \dot{z})\}$ is dense below p.
- $p \Vdash \varphi \wedge \psi$
- Note that $p \Vdash \dot{x} = \dot{y}$ is the same as $p \Vdash \dot{x} \subseteq \dot{y}$ and $p \Vdash \dot{y} \subseteq \dot{x}$.

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Remark.

- Formally, $p \Vdash \dot{x} \subseteq \dot{y}$ and $p \Vdash \dot{x} \in \dot{y}$ are circular definitions, and thus are we define them recursively.
- All the clauses, except for the existential one, use only absolute notions. Being dense below a condition is absolute.
- When relativising to $\mathcal{M}$, the key difference for the definition of forcing an existential is $(p \Vdash \exists x \varphi(x))^{\mathcal{M}}$, the $\exists \dot{x} \in V^{\mathbb{P}}$ becomes $\exists \dot{x} \in \mathcal{M}^{\mathbb{P}}$.

Proposition 15.15. *For $p \in \mathbb{P}$, $\varphi \in \mathcal{FL}_{\mathbb{P}}$, and $\dot{x}_1, \dots, \dot{x}_n \in V^{\mathbb{P}}$, the following are equivalent:*

(i) $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$.

(ii) $\forall q \leqslant p.\, q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$.

(iii) $\neq \exists q \leqslant p.\, q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$

(iv) $\{r \mid r \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)\}$ *is dense below* p.

Proof.

(ii) $\Rightarrow$ (iii) Suppose $\forall q \leqslant p$, $q \Vdash \varphi$, and $\exists q \leqslant p$, $q \Vdash \neg\varphi$. Then there is no $r \leqslant q$ such that $r \Vdash \varphi$. So $q \nVdash \varphi$.

(iii) $\Rightarrow$ (iv) Suppose $\neg\exists q \leqslant p$, $p \Vdash \neg\varphi$. Take $q \leqslant p$, then $q \nVdash \neg p$. So fix $r \leqslant q$ such that $r \Vdash \varphi$. So the set is dense below p.

(i) $\Rightarrow$ (ii) By induction on formula complexity:

- For atomic formulas, let $\square \in \{\in, \subseteq\}$. Then $p \Vdash \dot{x} \square \dot{y}$ iff some set A (from the definition of the forcing relation) is dense below p. Take

$q \leqslant p$. Then A is dense below q, since if $s \leqslant q \leqslant p$, there is some $r \leqslant s$ such that $r \in A$. Thus $q \Vdash \dot{x} \Box \dot{y}$.

- Suppose the claim holds for φ and ψ.

  If $q \leqslant p$, then $p \Vdash \neg\varphi$ says $\neg \exists r \leqslant p. (r \Vdash \varphi)$. Therefore $\neg \exists r \leqslant q (r \Vdash \varphi)$. So $q \Vdash \neg\varphi$.

  If $p \Vdash \varphi \wedge \psi$, then $p \Vdash \varphi$ and $p \Vdash \psi$. So by our inductive hypothesis, $q \Vdash \phi$ and $q \Vdash \psi$. Therefore $q \Vdash \phi \wedge \psi$.

  If $p \Vdash \exists x. \varphi(x)$, then this says that some set A is dense below p. This implies that A is dense below q (by our inductive hypothesis), so $q \Vdash \exists x. \varphi(x)$.

(iv) $\Rightarrow$ (i) As with the previous case, we proceed by induction on formula complexity.

- We will first do atomic formulas. Let $\Box \in \{\in, \subseteq\}$ again. To prove that $p \Vdash \dot{x} \Box \dot{y}$, we need to show that some set A is dense below p. But now, we have that $\{r \mid r \Vdash \dot{x} \Box \dot{y}\}$ is dense below p (which it is, by assumption). Fix some $q \leqslant p$, then there is some $r \leqslant q$ such that $r \Vdash \dot{x} \Box \dot{y}$. So there is some $s \leqslant r \leqslant q \leqslant p$ such that $s \in A$. Thus A is dense below p.

- The proof for $p \Vdash \exists x \varphi(x)$ is the same.

  Next, suppose that $\{r \mid r \Vdash \varphi \wedge \psi\}$ is dense below p. Since $r \Vdash \varphi \wedge \psi$ iff $r \Vdash \psi$ and $r \Vdash \varphi$. By the inductive hypothesis, $p \Vdash \varphi$ and $p \Vdash \psi$.

  Finally, suppose $\{r \mid r \Vdash \neg\varphi\}$ is dense below p. Fix $q \leqslant p$, and suppose for a contradiction that $q \Vdash \varphi$. Then by (i) $\Rightarrow$ (iii), there is no $r \leqslant q$ such that $r \Vdash \neg\varphi$, this contradicts that $\{r \mid r \Vdash \neg\varphi\}$ is dense below p.

□

Proposition 15.16. *For any $\mathbb{P}$, $p, q \in \mathbb{P}$, and name $\dot{a}, \dot{b} \in V^{\mathbb{P}}$:*

(i) $p \Vdash \dot{a} = \dot{a}$

(ii) $\langle q, \dot{b} \rangle \in \dot{a}$ *and* $p \leqslant q$ *then* $p \Vdash \dot{b} \in \dot{a}$.

(iii) If $\mathcal{M}$ is a transitive model of ZF *and $\mathbb{P} \in \mathcal{M}$, then for any φ, ψ, we have* $\{\langle q, \dot{x} \rangle \mid \langle q, \dot{x} \rangle \in \dot{a} \wedge (q \Vdash \varphi(\dot{x}))^{\mathcal{M}}\} \in \mathcal{M}$. *Also, we have that* $\{q \mid q \in \mathbb{P} \wedge q \Vdash (\psi(\dot{a}))^{\check{\mathcal{M}}}\} \in \mathcal{M}$. *So there are two ways of relativising, and they both work.*

(iv) $p \Vdash \varphi \vee \psi$ *iff* $\{q \leqslant p \mid q \Vdash \varphi$ *or* $q \Vdash \psi\}$.

(v) $p \Vdash \varphi \to \psi$ *iff for all* $\dot{x} \in V^{\mathbb{P}}$, $p \Vdash \varphi(\dot{x})$.

(vi) $p \Vdash \forall \dot{x} \varphi(\dot{x})$ *iff for all* $\dot{x} \in V^{\mathbb{P}}$, $p \Vdash \varphi(\dot{x})$.

(vii) *For every* φ, $\{p \in \mathbb{P} \,|\, p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ *is a dense open set.*

(viii) *There is no* p *and formula* φ *such that* $p \Vdash \varphi$ *and* $p \Vdash \neg\varphi$.

Proof. Example sheet 4 (probably). □

Lecture 15

Theorem 15.17 (Forcing Theorem)**.** *Suppose* $\mathcal{M}$ *is a transitive model of* ZF, $\mathbb{P} \in \mathcal{M}$, $\varphi(u)$ *is a formula, and* G *is* $\mathbb{P}$*-generic over* $\mathcal{M}$*. Then, for any* $\dot{x} \in \mathcal{M}^{\mathbb{P}}$:

(i) *If* $p \in G$ *and* $(p \Vdash \varphi(\dot{x}))^{\mathcal{M}}$, *then* $\mathcal{M}[G] \vDash \varphi(\dot{x}^G)$.

(ii) *If* $\mathcal{M}[G] \vDash \varphi(\dot{x}^G)$ *then* $\exists p \in G.\, (p \Vdash \varphi(\dot{x}))^{\mathcal{M}}$.

Proof. Proceed by induction on the complexity of formulas. Note that we need to work with $(p \Vdash \varphi(\bar{v}))^{\mathcal{M}}$, i.e. everything is relatizied, but since relativization and parameters are only important for statements with existential quantifiers, we will suppress them if there aren't any existential quantifiers.

Let $\Psi(\varphi)$ be the claim that for any $\dot{x} \in \mathcal{M}^{\mathbb{P}}$:

(i) If $p \in G$ and $(p \Vdash p(x))^{\mathcal{M}}$, then $\mathcal{M}[G] \vDash \varphi(\dot{x}^G)$.

(ii) If $M[G] \vDash \varphi(\dot{x}^G)$ then $\exists p \in G(p \Vdash \varphi(x))^{\mathcal{M}}$.

- So $\Psi(\varphi) \Rightarrow \Psi(\neg\varphi)$.
  - (i) Suppose $p \in G$, $p \Vdash \neg\varphi$. Suppose for a contradiction $\mathcal{M}[G] \vDash \varphi$, i.e. $\varphi^{\mathcal{M}[G]}$ holds. Then by $\Psi(\varphi)$, fix $q \in G$ such that $q \Vdash \varphi$. Since G is a filter, fix $r \leqslant p, q$. So $r \Vdash \varphi$, contradicting $p \Vdash \neg\varphi$. Thus $\neg(\varphi^{\mathcal{M}[G]})$. So $(\neg\varphi)^{\mathcal{M}[G]}$. That is, $M[G] \vDash \neg\varphi$.
  - (ii) Suppose that $M[G] \vDash \neg\varphi$, and let $D = \{p \in \mathbb{P} : p \Vdash \varphi \vee p \Vdash \neg\varphi\}$. D is dense, since if $q \nVdash \varphi$, then by definition there s some $p \leqslant q$ such that $p \Vdash \neg\varphi$. Then $p \in D$. So fix $p \in G \cap D$. If $p \Vdash \varphi$ then, by $\Psi(\varphi)$, $M[G] \vDash \varphi$. So $p \Vdash \neg\varphi$.
- $\Psi(\varphi) \wedge \Psi(\psi) \Rightarrow \Psi(\varphi \wedge \psi)$.
  - (i) Suppose $p \Vdash \varphi \wedge \psi$. Then $p \Vdash \varphi$ and $p \Vdash \psi$. Also $p \in G$. Since $\Psi(\varphi)$ and $\Psi(\psi)$ hold, $M[G] \vDash \varphi$ and $M[G] \vDash \psi$. So $M[G] \vDash \varphi \wedge \psi$.
  - (ii) Suppose $M[G] \vDash \varphi \wedge \psi$. Then $M[G] \vDash \varphi$ and $M[G] \vDash \psi$. Since $\Psi(\varphi)$ and $\Psi(\psi)$ hold, fix $p_1, p_2 \in G$ such that $p_1 \Vdash \varphi$ and $p_2 \Vdash \psi$. Since G is a filter, fix $r \leqslant p_1, p_2$. Then $r \Vdash \varphi \wedge \psi$.

- $\Psi(\varphi(\dot{x})) \Rightarrow \Psi(\exists x \varphi(x))$.

  1. Suppose $(p \Vdash \exists x \varphi(x))^{\mathcal{M}}$, $p \in G$. Let

  $$D = (\{q \leqslant p \,|\, \exists \dot{x} \in V^{\mathbb{P}}(q \Vdash \varphi(\dot{x}))\})^{\mathcal{M}}.$$
  $$= \{q \leqslant p \,|\, \exists \dot{x} \in \mathcal{M}^{\mathbb{P}}(q \Vdash \varphi(x))^{\mathcal{M}}\} \in \mathcal{M}$$

     By definition, D is dense. So since G is generic, fix $q \in G \cap D$. Then fix $\dot{x} \in \mathcal{M}^{\mathbb{P}}$ such that $(q \Vdash \varphi(\dot{x}))^{\mathcal{M}}$. So since $\Psi(\varphi(\dot{x}))$, $M[G] \vDash \varphi(\dot{x}^G)$. Therefore $M[G] \vDash \exists x.\, \varphi(x)$.

  2. Suppose $M[G] \vDash \exists x.\, \varphi(x)$. Fix $\dot{x} \in \mathcal{M}^{\mathbb{P}}$. such that $M[G] \vDash \varphi(\dot{x}^G)$. Since $\Psi(\varphi(x))$, there eixsts $p \in G$ such that $(p \Vdash \varphi(\dot{x}))$. Thus $\{q \leqslant p \,|\, (q \Vdash \varphi(\dot{x}))^{\mathcal{M}}\}$ is dense. Then, by definition, $(p \Vdash \exists x.\, \varphi(x))^{\mathcal{M}}$.

- $\Psi(x = y)$ We will prove next lecture, and we will assume it for now.

- $\Psi(x \in y)$

  (i) Suppose $p \Vdash \dot{x} \in \dot{y}$, $p \in G$. Let $D = \{q \leqslant p \,|\, \exists \langle r, \dot{z} \rangle \in \dot{y}(q \leqslant r \wedge q \Vdash \dot{x} = \dot{z})\}$. By definition, D is dense. Fix $q \in G \cap D$. Since $q \in D$, fix $\langle r, \dot{z} \rangle \in \dot{y}$ such that $q \leqslant r$ and $q \Vdash \dot{x} = \dot{z}$. Since $q \in G$, assuming that $\Psi(x = y)$, $\mathcal{M}[G] \vDash \dot{x}^G = \dot{z}^G$. Since G is a filter, $q \leqslant r$, $r \in G$, and so $\dot{z}^G \in \dot{y}$. So $M[G] \vDash \dot{x}^G \in \dot{y}^G$.

  (ii) Suppose $M[G] \vDash \dot{x}^G \in \dot{y}^G$. Then fix $\langle r, \dot{z} \rangle \in \dot{y}$ such that $r \in G$ and $\dot{z}^G = \dot{x}^G$. Now by $\Psi(x = y)$, there is some $q \in G$ such that $q \Vdash \dot{x} = \dot{z}$. Since G is a filter, fix $p \in G$ such that $p \leqslant q, r$. Finally, $p \Vdash \dot{z} \in \dot{y}$ and $p \Vdash \dot{x} = \dot{z}$, so for all $s \leqslant p$, $s \leqslant r$, and $s \Vdash \dot{x} = \dot{z}$. This gives that D is dense below p. hence $p \Vdash \dot{x} \in \dot{y}$.

□

Corollary 15.18. *Suppose $\mathcal{M}$ is a countable transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$, and $\varphi(u)$ is a formula. Then for any name $\dot{x}$:*

$$(p \Vdash \varphi(\dot{x}))^{\mathcal{M}} \Leftrightarrow \textit{for any } \mathbb{P}\textit{-generic filter } G \textit{ over } M \textit{ with } p \in G,\ \mathcal{M}[G] \vDash \varphi(\dot{x}^G)$$

Proof.

($\Rightarrow$) This is just clause (i) of the Forcing Theorem.

($\Leftarrow$) Suppose $(p \nVdash \varphi(\dot{x}))^{\mathcal{M}}$. Then there is some $q \leqslant p$ such that $(q \Vdash \neg\varphi(\dot{x}))^{\mathcal{M}}$. Now, let G be a $\mathbb{P}$-generic filter over $\mathcal{M}$ such that $q \in G$. Then, since G is upwards closed, $p \in G$. However, $M[G] \vDash \varphi$, but since $q \in G$, then by the Forcing Theorem, we get $M[G] \vDash \neg\varphi$. This leads to a contradiction, since then there is some p which forces φ and $\neg\varphi$ which contradicts part (viii) of proposition 15.16.

□

Definition 15.19 (Star forcing, $\Vdash^*$)**.** Suppose $\mathcal{M}$ is a countable transitive model of ZF, $\mathbb{P} \in \mathcal{M}$, $\dot{x}_1, \ldots \dot{x}_n \in \mathcal{M}^{\mathbb{P}}$, $p \in \mathbb{P}$, $\varphi(v_0, \ldots, v_n)$ a formula. Then define $\Vdash^*_{\mathbb{P},\mathcal{M}}$ by $p \Vdash^*_{\mathbb{P},\mathcal{M}} \varphi(\dot{x}_1, \ldots, \dot{x}_n)$ iff $M[G] \vDash \varphi(\dot{x}_1^G, \ldots, \dot{x}_n^G)$ for all $G \subseteq \mathbb{P}$ such that $p \in G$ and G is a $\mathbb{P}$-generic filter.

Lecture 16

Lemma 15.20. *If $\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$, $G \subseteq \mathbb{P}$ is generic, then $\mathcal{M}[G] \vDash$ Sep.*

Proof. Let $\varphi(x, v)$ be a formula, $\mathrm{Fr}(\varphi) \subseteq \{x, v\}$. Then it suffices to prove that for any $a, v \in \mathsf{M}[G]$, $b = \{x \in a \mid \mathcal{M}[G] \vDash \varphi(x, v)\} \in \mathcal{M}[G]$. Fix names $\dot{a}, \dot{v} \in \mathcal{M}^{\mathbb{P}}$ such that $\dot{a}^G = a$ and $\dot{v}^G = v$.

Any element of $\dot{a}^G$ is of the form $\dot{x}^G$ when $\langle p, \dot{x} \rangle \in \dot{a}$ and $p \in G$. Thus

$$b = \{\dot{x}^G \mid \exists p. (\langle p, \dot{x} \rangle \in \dot{a} \wedge p \in G \wedge M[G] \vDash \varphi(\dot{x}^G, \dot{a}^G)\}$$

. Let $\dot{b} = \{\langle p, \dot{x} \rangle \mid \langle p, \dot{x} \rangle \in dota \wedge (p \Vdash \varphi(\dot{x}, \dot{v}))^{\mathcal{M}}\} \in \mathcal{M}$. Thus $\dot{b}^G \in \mathcal{M}[G]$.

So

$$\begin{aligned} x \in \dot{b}^G &\Leftrightarrow \text{there is some } \mathbb{P}\text{-name } \dot{x} \in \mathcal{M} \text{ such that } \dot{x}^G = x,\ \langle p, \dot{x} \rangle \in \dot{a},\ (p \Vdash \varphi(\dot{x}, \dot{v}))^{\mathcal{M}}. \\ &\Leftrightarrow x \in \dot{a}^G \text{ and } M[G] \vDash \varphi(x, v) \\ &\Leftrightarrow x \in b \end{aligned}$$

□

Lemma 15.21. *$\mathcal{M}$ is a transitive model of* ZF*, $\mathbb{P} \in \mathcal{M}$, $G \subseteq \mathbb{P}$ is generic. Then $M[G] \vDash$ collection. It's easier to show that it models collection than replacement.*

Let $\varphi(x, y, v)$ be a formula with $\mathrm{Fr}(\varphi) \subseteq \{x, y, v\}$. Fox $a, v \in \mathcal{M}[G]$ with names $\dot{a}, \dot{v} \in \mathcal{M}^{\mathbb{P}}$.

Suppose that $\mathcal{M}[G] \vDash \forall x \in a\, \exists y,\ \varphi(x, y, v)$.

Then we claim that $\exists b \in \mathcal{M}[G]$ such that $\mathcal{M}[G] \vDash \forall x \in a\, \exists y \in b.\, \varphi(x, y, v)$. To prove this claim, let $C = \{\langle p, \dot{x} \rangle \mid p \in \mathbb{P},\ \dot{x} \in \mathrm{ran}(\dot{a}),\ \exists \dot{y} \in \mathcal{M}^{\mathbb{P}} (p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^{\mathcal{M}}\}$.

Now, for all $\langle p, \dot{x} \rangle \in C$. $\exists \dot{y} \in \mathcal{M}^{\mathbb{P}}(p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^{\mathcal{M}}$.

Note that there may be class many $\dot{y}$ such that $p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v})$. Using collection in $\mathcal{M}$, there is a set $B \in \mathcal{M}$, $B \subseteq \mathcal{M}^{\mathbb{P}}$ such that $\forall \langle p, \dot{x} \rangle \in C, \exists doty \in B. (p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^{\mathcal{M}}$. Finally, set $\dot{b} = \{\langle \mathbb{1}, \dot{y} \rangle \,|\, \dot{y} \in B\} \in \mathcal{M}^{\mathbb{P}}$. Now to show that $\dot{b}^G$ suffices, fix $x \in a$. We can find $\langle q, \dot{x} \rangle \in \dot{a}$ such that $q \in G$ and $\dot{x}^G = x$. By our assumption, $\mathcal{M}[G] \Vdash \exists y \, \varphi(x, y, v)$. So fix $\dot{z}^G$ such that $\mathcal{M}[G] \vDash \varphi(x, \dot{z}^G . v)$.

Then by the forcing theorem, fix $p \in G$ such that $(p \Vdash \varphi(\dot{x}, \dot{z}, \dot{v}))^{\mathcal{M}}$. From this, it follows that $\langle p, x \rangle \in C$. So we can fix $\dot{y} \in B$ such that $(p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^{\mathcal{M}}$. Therefore $\langle \mathbb{1}, \dot{y} \rangle \in \dot{b}$. Since $\mathbb{1} \in G$, $\dot{y}^G \in \dot{b}^G$. Finally, by the forcing theorem, $\mathcal{M}[G] \vDash \dot{y}^G \in \dot{b}^G \wedge \varphi(\dot{x}^G \dot{y}^G, v)$. Note that we haven't used power set here, so if $\mathcal{M} \vDash \mathsf{ZF}^-$, then $\mathcal{M}[G] \vDash \mathsf{ZF}^-$. ☐

Lecture 17 It only remains to show that if $\mathcal{M} \vDash \mathsf{ZFC}$, then $\mathcal{M}[G] \vDash$ power set.

Recall:

(a) for all $\langle q_1, z_1 \rangle \in \dot{x}$, $\{r \leqslant p \,|\, r \leqslant q_1 \to \exists \langle q_2, z_2 \rangle \in y. (r \leqslant q_2 \wedge r \Vdash \dot{z}_1 = \dot{z}_2)\}$ is dense below p.

(b) For all $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$, $\{r \in p \,|\, r \leqslant q_2 \to \exists \langle q_1, \dot{z}_1 \rangle \in \dot{x}. (r \leqslant q_1 \wedge r \Vdash \dot{z}_1 = \dot{z}_2)\}$.

Then, continuing to prove the forcing theorem, it remains to prove:

Lemma 15.22 (F). *or any $\dot{x}, \dot{y} \in \mathcal{M}^{\mathbb{P}}$,*

(i) If $p \in G$ and $(p \Vdash \dot{x} = \dot{y})^{\mathcal{M}}$, then $\mathcal{M}[G] \vDash \dot{x}^G = \dot{y}^G$.

(ii) If $\mathcal{M}[G] \vDash \dot{x}^G = \dot{y}^G$, then $\exists p \in G (p \Vdash \dot{x} = \dot{y})^{\mathcal{M}}$.

Proof. We proceed by transfinite induction on the pair $(\dot{x}, \dot{y})$ when ordered lexicographically.

(i) Suppose $p \Vdash \dot{x} = \dot{y}$, $p \in G$. To prove that $\mathcal{M}[G] \vDash \dot{x}^G = \dot{y}^G$, it suffices to show that $\mathcal{M}[G] \vDash \dot{x}^G \subseteq \dot{y}^G$, and then we obtain equality by symmetry.

Any element of $\dot{x}^G$ must be of the form $\dot{z}_1^G$ where $\langle q_1, \dot{z}_1 \rangle \in \dot{x}$, and $q_1 \in G$. Since G is a filter, fix $s \in G$ such that $s \leqslant p, \dot{q}_1$. Since $s \leqslant p$, $s \Vdash \dot{x} = \dot{y}$. So

$$\{r \leqslant s \,|\, r \leqslant q_1 \to \exists \langle q_2, \dot{z}_2 \rangle \in y. (r \leqslant q_2 \wedge r \Vdash \dot{z}_1 = \dot{z}_2)\} \cap G \neq \emptyset.$$

Fix r in this intersection. Then $r \leqslant s \leqslant q_1$. So fix $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$ such that $r \leqslant q_2 \wedge r \Vdash \dot{z}_1 = \dot{z}_2$. Since G is generic, $q_2 \in G$. Therefore $\dot{z}_1^G \in \dot{y}^G$. Then by induction, since $r \in G$, $\mathcal{M}[G] \vDash \dot{z}_1^G = \dot{z}_2^G$. Thus, $\dot{z}_1^G \in \dot{y}^G$. So we have proven that $\mathcal{M} \vDash \dot{x}^G \subseteq \dot{y}^G$.

(ii) Suppose $\mathcal{M}[G] \vDash \dot{x}^G = \dot{y}^G$. Then define D to be the set of conditions $r \in \mathbb{P}$ such that at least one of the following hold

(0) $r \Vdash \dot{x} = \dot{y}$.

(a') $\exists \langle q_1, \dot{z}_1 \rangle \in \dot{x}. (r \leqslant q_1 \wedge \forall \langle q_2, \dot{z}_2 \rangle \in \dot{y}, \forall s \in \mathbb{P}((s \leqslant q_2 \wedge s \Vdash \dot{z}_1 = \dot{z}_2) \rightarrow s \perp r)$.

(b') $\exists \langle q_2, \dot{z}_2 \rangle \in \dot{y}(r \leqslant q_2 \wedge \forall \langle q_1, \dot{z}_1 \rangle \in \dot{x} \forall s \in \mathbb{P}((s \leqslant q_1 \wedge s \Vdash \dot{z}_1 = \dot{z}_2) \rightarrow s \perp r))$.

Note by separation in $\mathcal{M}$ and absoluteness results, $D \in \mathcal{M}$. Then we claim:

Claim: D is dense.

To show this, fix $p \in \mathbb{P}$ and suppose $p \nVdash \dot{x} = \dot{y}$. Then at least one of (a) or (b) in the definition of $\Vdash$ for equality fails. Suppose it were (a), then we can fix $\langle q_1, \dot{z}_1 \rangle \in \dot{x}$ such that $\{r \leqslant p \,|\, r \leqslant q_1 \rightarrow \exists \langle q_2, \dot{z}_2 \rangle \in \dot{y}(r \leqslant q_2 \wedge r \Vdash \dot{z}_1 = \dot{z}_2)\}$ is not dense below p. This means we can fix $s \leqslant p$ such that

$$\forall r \leqslant s. (r \leqslant q_1 \wedge \forall \langle q_2, \dot{z}_2 \rangle \in \dot{y}. \neg(r \Vdash \dot{z}_1 = \dot{z}_2 \wedge r \leqslant q_2)) \qquad (*)$$

. Observe that this gives $s \leqslant q_1$. Now, if we fix $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$, $r \leqslant q_2$, and $r \Vdash \dot{z}_1 = \dot{z}_2$, then it must be the case that $r \perp s$, as any common extension of s and r would contradict $(*)$. Thus, $s \leqslant p$ and s satisfies (a').

Since D is dense, D is dense below $p \in G$, G is $\mathbb{P}$-generic, fix $r \in G \cap D$. We shall show r satisfies (0), which finishes the proof. Suppose for contradiction that r satisfies (a'), then we can fix $\langle q_1, z_1 \rangle$ satisfying the statement of (a'). Since $r \in G$ and $r \leqslant q_1$, we must have that $q_1 \in G$ by upwards closure of filters. Therefore $\mathcal{M}[G] \vDash z_1^G \in \dot{x}^G = \dot{y}^G$. So we can fix $\langle q_2, \dot{z}_2 \rangle \in y$ such that $q_2 \in G$ and $\mathcal{M}[G] \vDash \dot{z}_1^G = \dot{z}_2^G$. By the inductive hypothesis, fix $p' \in G$ such that $p' \Vdash \dot{z}_1 = \dot{z}_2$. Now since G is a filter, and $p', q_2 \in G$, we can find $s \in G$ so that $s \leqslant p', q_2$. Then $s \Vdash \dot{z}_1 = \dot{z}_2$, so by (a'), $s \perp r$. But now $s \in G$ and $r \in G$ so they must be compatible. Contradiction. Therefore (a') doesn't hold for r. By symmetry, neither can (b'). Then since $r \in D$ and one of (0), (a'), and (b') must hold, (0) must hold.

□

Recall from last lecture that if $\mathcal{M} \vDash \mathsf{ZF}^-$, and G is $\mathbb{P}$-generic, then $\mathcal{M}[G] \vDash \mathsf{ZF}^-$.

Lemma 15.23. *If $\mathcal{M} \vDash \mathsf{ZF}$, then $\mathcal{M}[G] \vDash \mathsf{ZF}$.*

Proof. It only remains to prove power set. For this, it suffices to show that if $a \in \mathcal{M}[G]$, then $\mathcal{P}(a) \cap \mathcal{M}[G] = \{x \in \mathcal{M}[G] \mid x \subseteq a\} \subseteq b$ for some $b \in \mathcal{M}[G]$.

Fix $a \in \mathcal{M}[G]$ with corresonding name $\dot{a}$. Then define

$$S = \{x \in \mathcal{M}^{\mathbb{P}} \mid \operatorname{ran}(x) \subseteq \operatorname{ran}(\dot{a})\} = \mathcal{P}(\mathbb{P} \times \operatorname{ran}(a))^{\mathcal{M}}.$$

Let $\dot{b} = \{\langle \mathbb{1}, \dot{x}\rangle \mid x \in S\} \in \mathcal{M}^{\mathbb{P}}$. Let $c \in \mathcal{P}(a) \cap \mathcal{M}[G]$, we need to show that $c \in \dot{b}^G$. Let $\dot{c} \in \mathcal{M}^{\mathbb{P}}$ be such that $c = \dot{c}^G$. Then let

$$\dot{x} = \{\langle p, \dot{z}\rangle \mid \dot{z} \in \operatorname{ran}(\dot{a}) \wedge (p \Vdash \dot{z} \in \dot{c})^{\mathcal{M}}\} \in S.$$

Claim: $\dot{x}^G = \dot{c}^G = c$.

($\dot{x}^G \subseteq c$) To show this, fix $\dot{z}^G \in \dot{x}^G$. Then by definition, we can fix $p \in G$ such that $\langle p, \dot{z}\rangle \in x$. From this it follows that $z \in \operatorname{ran}(\dot{a})$ and $p \Vdash \dot{z} \in \dot{c}$. Since $p \in G$, by the Forcing Theorem, $\mathcal{M}[G] \vDash \dot{z}^G \in c$.

($c \subseteq \dot{x}^G$) Since $\mathcal{M}[G] \vDash c \subseteq \dot{a}^G$, we know every element of c is of the form $\dot{z}^G$, for some pair $\langle q, \dot{z}\rangle \in a$ with $q \in G$. Also, if $\mathcal{M}[G] \vDash \dot{z}^G \in c$, then by the Forcing theorem, we can fix $p \in G$ which forces this, so $p \Vdash \dot{z} \in c$. Thus $\langle p, z\rangle \in x$. So $\dot{z}^G \in \dot{x}^G$.

□

Lemma 15.24. $\mathcal{M} \vDash \mathsf{ZFC}^- \Rightarrow \mathcal{M}[G] \vDash \mathsf{ZFC}^-$. *(And obviously also for* ZFC*, by the above).*

Proof. To show this, we will show well-ordering holds. It suffices to show that any $a \in \mathcal{M}[G]$ can be well-ordered in $\mathcal{M}[G]$. (There is definitely one in the universe, since a is countable, but we need to show that it exists in $\mathcal{M}[G]$). Using well-ordering in $\mathcal{M}$, list the elements of $\operatorname{ran}(a)$ as $\{\dot{x}_\alpha : \alpha < \delta\}$ for some $\delta \in \mathrm{Ord}$.

Then let $\dot{f} = \{\langle \mathbb{1}, |(\rangle\langle(|\, \check{\alpha}, x_\alpha)\rangle \mid \alpha < \delta\} \in \mathcal{M}^{\mathbb{P}}$. Then in $\mathcal{M}[G]$, $\dot{f}^G = \{\langle \alpha, \dot{x}_\alpha^G\rangle \mid \alpha < \delta\}$. So $\dot{f}^G$ is a function with domain δ and $a \subseteq \operatorname{ran}(zdotf^G)$. Therefore, define a well-order on a by saying $x < y$ if and only if

$$\min\{\alpha < \delta \mid \dot{f}^G(\alpha) = \dot{x}\} < \min\{\alpha < \delta \mid \dot{f}^G(\alpha) = y\}.$$

Which finishes the proof. □

Remark.

1. $\dot{f}^G$ may not be injective, we could have $\dot{x}_\alpha^G = \dot{x}_\beta^G$.

2. $\operatorname{ran}(\dot{f}^G)$ may not equal a. (Elements of $\dot{a}$ are $\langle p, x_\alpha\rangle$, if $p \notin G$, we may not have $x_\alpha^G \in a$).

3. For Power set it sufficed to find a set of names which contained enough names to represent all possible subsets of a.
4. $\mathcal{M}[G] \vDash \varphi$ should be considered as a ternary relation, it is a statement about $\mathcal{M}$, about G, and about φ. We could have two generic filters G and H, such that $\mathcal{M}[G] \vDash \varphi$ and $\mathcal{M}[H] \vDash \neg\varphi$.
5. We were careful to write $(p \Vdash \varphi)^{\mathcal{M}}$ with the relativisation to $\mathcal{M}$. However, we will drop this relativisation when it is clear.

Lecture 18

Lemma 15.25. *Let φ and ψ be two formulas in our forcing language. Let $\mathcal{M}$ be a countable transitive model of* ZFC*, and let $\mathbb{P} \in \mathcal{M}$. Then for any $p \in \mathbb{P}$, and $x \in \mathcal{M}^{\mathbb{P}}$:*

(i) If ZF $\vdash \forall v. (\varphi(v) \to \psi(v))$ *then*

$$(p \Vdash \varphi(\dot{x}))^{\mathcal{M}} \Rightarrow (p \Vdash \psi(\dot{x}))^{\mathcal{M}}$$

(ii) If ZFC $\vdash \forall v. (\varphi(v) \leftrightarrow \psi(v))$*, then*

$$(p \Vdash \varphi(\dot{x}))^{\mathcal{M}} \Leftrightarrow (p \Vdash \psi(\dot{x}))^{\mathcal{M}}$$

Proof. Suppose ZFC $\vdash \forall v. (\varphi(v) \to \psi(v))$. and $(p \Vdash \varphi(\dot{x}))^{\mathcal{M}}$. Since $\mathcal{M}$ is countable, let G be a $\mathbb{P}$-generic filter over $\mathcal{M}$ such that $p \in G$. By the forcing theorem, $\mathcal{M}[G] \vDash \varphi(\dot{x}^G)$. Then since $\mathcal{M}[G] \vDash$ ZFC, we must have $\mathcal{M}[G] \vDash \psi(\dot{x}^G)$. By the forcing theorem, since true for all generics containg p, $(p \Vdash \psi(\dot{x}))^{\mathcal{M}}$. □

16 Forcing and Independence Results

We want to show Cons(ZFC $+ V \neq L$) assuming Cons(ZFC). Before we do that, we need one theorem:

Theorem 16.1. *Let $\mathcal{M}$ be a countable transitive model of* ZFC*, then there is a countable transitive model $N \supseteq M$ of* ZFC *such that $N \vDash$* ZFC $+ V \neq L$*.*

Proof. Let $\mathcal{M}$ be a countable transitive model of ZFC, and $\mathbb{P} \in \mathcal{M}$ be any atomless forcing poset (i.e. having no minimal element). The obvious example of this is $\mathrm{Fn}(\omega, 2)$. Since $\mathcal{M}$ is countable, let G be $\mathbb{P}$-generic over $\mathcal{M}$. Since $\mathbb{P}$ is atomless, $G \notin \mathcal{M}$. So $M \subsetneq M[G]$ is a model of ZFC. By the generic model theorem, $\mathrm{Ord} \cap \mathcal{M} = \mathrm{Ord} \cap \mathcal{M}[G]$. Therefore, $L_{\mathrm{Ord} \cap \mathcal{M}} = \mathcal{L}_{\mathrm{Ord} \cap M[G]} = L^{\mathcal{M}} \subseteq \mathcal{M} \subsetneq \mathcal{M}[G]$. Therefore $M[G] \neq L_{\mathrm{Ord} \cap \mathcal{M}[G]} = L^{\mathcal{M}[G]}$. So $(V \neq L)^{\mathcal{M}[G]}$. $\square$

This doesn't yet give us the consistency proof, since we had to assume the existence of a countable transitive model. Now we prove:

Theorem 16.2. $\mathrm{Con}(\mathsf{ZFC}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + V \neq L)$. *Therefore* $\mathsf{ZFC} \nvdash V = L$.

Proof. Assume that $\mathsf{ZFC} + V \neq L$ gives rise to a contradiction. Then, from a finite set of axioms Γ of $\mathsf{ZFC} + V \neq L$, we can find ψ such that $\Gamma \vdash \psi \wedge \neg\psi$. By following the previous proofs, there is a finite set of axioms Λ of ZFC such that: $\mathsf{ZFC} \vdash$ if there is a countable transitive model for Λ, then there is a countable transitive model for $\Gamma + V \neq L$.

Λ should be sufficient to:

- Prove the basic properties of forcing and constructibility.
- Prove the necessary absoluteness facts (e.g. "finite", "partial order").
- Prove facts about forcing (the forcing theorem)
- Finally, to be such that ZFC can prove that if $\mathcal{M}$ is a countable transitive model of Λ, $\mathbb{P} \in \mathcal{M}$. G is $\mathbb{P}$-generic over $\mathcal{M}$, then $\mathcal{M}[G] \vDash \Gamma$.

Finally, by reflection, since Λ is finite, and a subset of ZFC, there is a countable transitive model of Λ. Therefore there is a countable transitive model of $\Gamma + V \neq L$. But $\Gamma \vdash \psi \wedge \neg\psi$. So $\mathcal{N} \vDash \psi \wedge \neg\psi$. So $\psi \wedge \neg\psi)^{\mathcal{N}}$. So by relativisation, $\mathsf{ZFC} \vdash \psi^{\mathcal{N}} \wedge \neg\psi^{\mathcal{N}}$. So $\neg\,\mathrm{Con}(\mathsf{ZFC})$. $\square$

Remark. Someone showed that this can be done in Isabelle (very recently). This is called "The formal verification of the ctm approach to forcing."

To get $\mathrm{Con}(\mathsf{ZFC}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + \neg CH)$ takes ZC plus 21 instances of replacement.

16.1 Cohen Forcing

Fix $\mathcal{M}$ to be a countable transitive model of ZFC. Recall that for $I, J \in \mathcal{M}$:

- $\mathrm{Fn}(I, J) = \{p \mid |p| < \omega,\, p \text{ is a function}, \mathrm{dom}(p) \subseteq I,\, \mathrm{ran}(p) \subseteq J\}$.
- Also $(\mathrm{Fn}(I, J), \supseteq, \emptyset)$ is a forcing poset.
- $\mathrm{Fn}(I, J) \in M$.
- $\mathrm{Fn}(I, J)$ has the countable chain condiion iff $I \neq \emptyset$ or J is countable.

- The sets $D_i = \{q \in \text{Fn}(I, J) \mid i \in \text{dom}(q)\}$, and $R_j = \{q \in \text{Fn}(I, J) \mid j \in \text{ran}(q)\}$ are dense for all $i \in I$, $j \in J$.

Now, suppose that $G \subseteq \text{Fn}(I, J)$ ws $\text{Fn}(I, J)$-generic over $\mathcal{M}$. Since G is a filter, if $p, q \in G$, then $p \cap q \in G$. So if $p, q \in G$, then p and q agree on the intersection of their domains.

Since wherever they overlap, they agree, we can take $f_G = \cup G$. Then f_G is a function with $\text{dom}(f_G) \subseteq I$, and $\text{ran}(f_G) \subseteq J$. Note that a name for f_G is

$$\dot{f} = \{\langle p, |\langle\rangle\langle(|\,\check{i}, \check{j})\rangle \mid p \in \mathbb{P}, \langle i, j\rangle \in p\}.$$

Since D_i and R_j are dense for all i, j $G \cap D_i \neq$ *emptyset* so $i \in \text{dom}(f_G)$. Thus

Proposition 16.3. *If $G \subseteq \text{Fn}(I, J)$ is $\text{Fn}(I, J)$-generic over $\mathcal{M}$ and $I, J \neq \emptyset$, then $\mathcal{M}[G] \vDash f_G : I \to J$ is a surjection.*

Proof. Omitted. □

Proposition 16.4. *Suppose that I, J are non-empty sets, at least one of which is infinite. Then $|\text{Fn}(I, J)| = \max\{|I|, |J|\}$.*

Proof. Each $p \in \text{Fn}(I, J)$ is a finite function, so $\text{Fn}(I, J) \subseteq {}^{<\omega}(I \times J)$. Then:

$$\begin{aligned} |\text{Fn}(I, J)| &\leqslant |(I \times J)^{<\omega}| \\ &= |I \times J| \\ &= \max\{|I|, |J|\}. \end{aligned}$$

For the reverse, fix $i_0 \in I$, and $j_0 \in J$. Then $\{\langle i_0, j\rangle \mid j \in J\} \cup \{\langle i, j_0\rangle \mid i \in I\}$ is a collection of $|I \cup J|$ many distinct elements of $\text{Fn}(I, J)$. Then $|I \cup J| = \max\{|I|, |J|\} \leqslant |\text{Fn}(I, J)|$. □

To provide a model where CH fails, consider $\text{Fn}(\omega_2^{\mathcal{M}} \times \omega, 2)$. Consider $f_G : \omega_2^{\mathcal{M}} \times \omega \to 2$. Let $g_\alpha : \omega \to 2$. be defined by

$$f_G(\alpha, n) = g_\alpha(n).$$

In order to get that $\mathcal{M}[G] \vDash \mathsf{ZFC} + \neg\mathsf{CH}$, it remains to:

- Prove that for $\alpha \neq \beta$, $g_\alpha \neq g_\beta$.
- Prove that $\omega_1^{\mathcal{M}[G]} = \omega_1$ and $\omega_2^{\mathcal{M}[G]} = \omega_2^{\mathcal{M}}$.

It will turn out that the countable chain condition (ccc) guarantees that all cardinals in $\mathcal{M}$ remain cardinals in $\mathcal{M}[G]$. This is not trivially true, consider the following examples:

Example. Let κ be an uncountable cardinal in $\mathcal{M}$. Consider $\mathrm{Fn}(\omega, \kappa)$. Then in $\mathcal{M}[G]$, $f_G : \omega \to \kappa$ is a surjection. Therefore $(\kappa \text{ is countable})^{\mathcal{M}[G]}$. The reason this broke was because $\mathrm{Fn}(\omega, \kappa)$ does not have the countable chain condition.

Lecture 19

Definition 16.5 (Preserving cardinals/cofinalities)**.** Let $\mathbb{P} \in \mathcal{M}$ be a forcing poset. Then we say that:

(i) $\mathbb{P}$ **preserves cardinals** iff for every generic filter $G \subseteq \mathbb{P}$ on $\mathcal{M}$

$$(\kappa \text{ is a cardinal})^{\mathcal{M}} \Leftrightarrow (\kappa \text{ is a cardinal})^{\mathcal{M}[G]}$$

for all $\kappa \in \mathrm{Ord}$.

(ii) $\mathbb{P}$ **preserves cofinalities** iff for every genric filter $G \subseteq \mathbb{P}$ over $\mathcal{M}$

$$\mathrm{cf}^{\mathcal{M}}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$$

for all limit ordinals $\gamma \in \mathrm{Ord} \cap \mathcal{M}$.

Remark.

- Being a cardinal is Π_1 definable, so downwards absolute.
- Finite and ω are absolute.

Also, remember that we showed the forcing $\mathrm{Fn}(\omega, \kappa)$ collapses κ, so there are forcings which do *not* preserve cardinals.

Lemma 16.6. *Suppose $\mathbb{P} \in \mathcal{M}$ is our forcing poset. Then*

(i) $\mathbb{P}$ preserves cofinalities iff for all $\mathbb{P}$-generic filters G for all limit cardinals β with $\omega < \beta < \mathrm{Ord} \cap \mathcal{M}$,

$$(\beta \textit{ is regular})^{\mathcal{M}} \to (\beta \textit{ is regular})^{\mathcal{M}[G]}.$$

(ii) If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.

Proof.

(i, $\Rightarrow$) Suppose $\mathbb{P}$ preserves cofinalities, and G is $\mathbb{P}$-generic. Fix β, $\omega < \beta < \mathrm{Ord} \cap \mathcal{M}$, a limit. Then if $(\beta \text{ is regular})^{\mathcal{M}}$, $\beta = \mathrm{cf}^{\mathcal{M}}(\beta) = \mathrm{cf}^{\mathcal{M}[G]}(\beta)$. So if β is regular in $\mathcal{M}$ then it is regular in $\mathcal{M}[G]$.

(i, $\Leftarrow$) Let γ be a limit ordinal with $\omega < \gamma < \mathrm{Ord} \cap \mathcal{M}$. Let $\beta = \mathrm{cf}^{\mathcal{M}}(\gamma)$. Then β is regular in $\mathcal{M}$, since the cofinality of an ordinal is always regular. Let $f \in \mathcal{M}$, $f : \beta \to \gamma$ be strictly increasing and cofinal. Then if β is uncountable in $\mathcal{M}$, then $(\beta \text{ is regular})^{\mathcal{M}[G]}$ holds, by assumption. Otherwise, $\beta = \omega$. Then $(\beta = \omega)^{\mathcal{M}[G]}$ so $(\beta \text{ is regular})^{\mathcal{M}[G]}$. Finally, since $f \in \mathcal{M}$, $f \in \mathcal{M}[G]$, so there is a strictly increasing cofinal map from β to γ in $\mathcal{M}[G]$. Thus, $\mathrm{cf}^{\mathcal{M}[G]}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\beta) = \beta = \mathrm{cf}^{\mathcal{M}}(\gamma)$, so cofinalities are preserved as required.

(ii) Suppose $\mathbb{P}$ preserves cofinalities and let κ be a cardinal in $\mathcal{M}$. One of three cases occur,

(a) $\kappa \leqslant \omega$, so $(\kappa \leqslant \omega)^{\mathcal{M}[G]}$, so $(\kappa \text{ is a cardinal})^{\mathcal{M}[G]}$.

(b) κ is regular. Then by (i), $(\kappa \text{ is regular})^{\mathcal{M}[G]}$. In particular, regular implies cardinal, so $(\kappa \text{ is a cardinal})^{\mathcal{M}[G]}$.

(c) $(\kappa \text{ is singular})^{\mathcal{M}}$. Then κ is the supremum of a set of regular cardinals. So since $\mathbb{P}$ preserves regular cardinals, every element of this set is regular in $\mathcal{M}[G]$. Therefore κ is the supremum of a set of cardinals in $\mathcal{M}[G]$, and thus a cardinal in $\mathcal{M}[G]$ (by a question on Example Sheet 1).

□

Lemma 16.7. *Let $A, B, \mathbb{P} \in \mathcal{M}$. Then if $(\mathbb{P}$ has the ccc$)^{\mathcal{M}}$, and G is $\mathbb{P}$-generic over $\mathcal{M}$. Then for any function $f \in \mathcal{M}[G]$ with $f : A \to B$, there is a fucntion $F \in \mathcal{M}$ with $F : A \to \mathcal{P}^{\mathcal{M}}(B)$ such that for all $a \in A$, $f(a) \in F(a)$, $|F(a)| \leqslant \aleph_0$.*

Proof. Suppose that $f : A \to B$ is our function in $\mathcal{M}[G]$. Since A, B are in $\mathcal{M}$, let $\check{A}, \check{B}$ be the canonical names. Let $\dot{f}$ be a name for f. Then, by the forcing theorem, there is a $p \in G$ such that

$$p \Vdash \dot{f} : \check{A} \to \check{B}$$

is a function.

Define $F : A \to \mathcal{P}^{\mathcal{M}}(B)$ by $F(a) = \{b \in B \mid \exists q \leqslant p. (q \Vdash \dot{f}(\check{a}) = \check{b})\}$. Then we prove that $F(a)$ has the desired properties.

Claim: $F \in \mathcal{M}$.

This is fine, since by the definability of the forcing relation, we have that

$F(a) \in \mathcal{M}$ for all $a \in \mathcal{M}$. So, since $A \in \mathcal{M}$, $F = \{\langle a, F(a)\rangle \mid a \in A\} \in M$ by replacement.

Observe: F is a function.

Claim: $f(a) \in F(a)$.

Suppose that $\mathcal{M}[G] \vDash f(a) = b$ for $b \in B$. Then by the forcing theorem, we can fix $q \in G$ such that $q \Vdash \dot{f}(\check{a}) = \check{b}$. Since G is a filter, fix $r \leqslant p, q$ with $r \in G$. Then r forces that $\dot{f}$ is a function, and that $\dot{f}(\check{a}) = \check{b}$. So $b \in F(a)$.

Claim: $|F(a)| \leqslant \aleph_0$.

Working in $\mathcal{M}$, using the axiom of choice in $\mathcal{M}$. For each $b \in F(a)$, choose $q_b \leqslant p$ such that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$. We shall show that $q_b \perp q_c$ for $b \neq c$, and then the countable chain condition will entail immediately that $|F(a)| \leqslant \aleph_0$.

Suppose not, then $r \leqslant q_b, q_c$ since they are compatible. Then $r \Vdash \dot{f} : \check{A} \to \check{B}$ is a function $\wedge\ \check{b} \neq \check{c} \wedge f(\check{a}) = \check{b} \wedge f(\check{a}) = \check{c}$. Now let H be a generic filter with $r \in H$. Then $r \leqslant p$ and $\mathcal{M}[H] \vDash f : A \to B$ is a function $\wedge$ $f(a) = b \wedge f(a) = c \wedge b \neq c$. This gives a contradiction. □

Theorem 16.8. *If $\mathbb{P} \in \mathcal{M}$ and ($\mathbb{P}$ has ccc)$^{\mathcal{M}}$. then $\mathbb{P}$ preserves cofinalities and hence cardinals.*

Proof. Using the previous lemma, it suffices to show that if $\omega < \beta < \mathrm{Ord} \cap \mathcal{M}$, and β is a limit, then if (β is regular)$^{\mathcal{M}}$ then (β is regular)$^{\mathcal{M}[G]}$. Suppose for a contradiction that β is regular in $\mathcal{M}$ but singular in $\mathcal{M}[G]$.

Then in $\mathcal{M}[G]$ fix a cofinal map $f : \alpha \to \beta$ for some $\alpha < \beta$. So in $\mathcal{M}$ there is some function $F : \alpha \to \mathcal{P}^{\mathcal{M}}(\beta)$ such that for all $\gamma \in \alpha$, $f(\gamma) \in F(\gamma)$, and $|F(\gamma)| \leqslant \aleph_0$. Then let

$$X = \bigcup_{\gamma < \alpha} F(\gamma).$$

Then we have that $X \subseteq \beta$, and X is the union of less than β-many countable sets. So $X \neq \beta$. But, f was a cofinal function, so:

$$\beta \bigcup_{\alpha < \gamma} f(\alpha).$$

But this means β is contained in X by the definition of our approximation map F. Therefore $\beta = X$. But this is a contradiction. □

16.2 The Failure of CH

Theorem 16.9. *Fix $\alpha < \mathrm{Ord} \cap \mathcal{M}$ and let $\kappa = (\aleph_\alpha)^{\mathcal{M}}$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$ generic over $\mathcal{M}$. Then $\mathcal{M}[G]$ contains a κ-length sequence of distinct elements of 2^ω. Hence,*

$$\mathcal{M}[G] \vDash \mathsf{ZFC} + 2^{\aleph_0} \geqslant \kappa = \aleph_\alpha.$$

Proof. Let $f = \cup G \in \mathcal{M}[G]$. Then we have already shown earlier that f is a function from $\kappa \times \omega$ onto 2. For $\beta < \kappa$, let $g_\beta : \omega \to 2$, $g_\beta = f(\beta, n)$.

Claim: $\alpha \neq \beta$, $g_\alpha \neq g_\beta$. Then define $E_{\alpha,\beta} \in \mathcal{M}$ by

$$E_{\alpha.\beta} = \{q \in \mathbb{P} \mid \exists n. (\langle \beta, n\rangle, \langle \alpha, n\rangle \in \mathrm{dom}(q) \wedge q(\langle \beta, n\rangle) \neq q(\langle \alpha, n\rangle))\}$$

Then to prove that $E_{\alpha,\beta}$ is dense, fix $p \in \mathbb{P}$. Since p is finite, tere is some m such that $\langle \beta, m\rangle, \langle \alpha, m\rangle \notin \mathrm{dom}(p)$.

Define $q \leqslant p$ by $q : \mathrm{dom}(p) \cup \{\langle \beta, m\rangle, \langle \alpha, m\rangle\} \to 2$ by

- $p(z) = q(z)$ if $z \in \mathrm{dom}(p)$.
- $q(\langle \beta, m\rangle) = 1$.
- $q(\langle \alpha, m\rangle) = 0$.

Then $q \in E_{\alpha,\beta}$. Since G is $\mathbb{P}$-generic, fix $q \in G \cap E_{\alpha,\beta}$. Then:

$$\begin{aligned} g_\beta(m) &= f(\beta, m) = q(\langle \beta, m\rangle) \\ &\neq q(\langle \alpha, m\rangle) = f(\alpha, m) = g_\alpha(m). \end{aligned}$$

for the chosen m.

Lecture 20 Finally, since $\mathbb{P}$ has the ccc in $\mathcal{M}$, $\mathbb{P}$ preserves cardinals, so $\kappa = (\aleph_\alpha^{\mathcal{M}[G]})$. To recap the previous lecture, we had that if $\mathcal{M}$ were a countable transitive model of ZFC, and $\alpha < \mathrm{Ord} \cap \mathcal{M}$, and $(\kappa = \aleph_\alpha)^{\mathcal{M}}$, $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Then letting G be $\mathbb{P}$-generic, and $\cup G : \kappa \times \omega \to 2$.

Then for $\gamma < \kappa$, we defined $g_\gamma : \omega \to 2$ so that $g_\gamma(n) = \cup G(\gamma, n)$. Then if $\gamma \neq \gamma'$, $g_\gamma \neq g_{\gamma'}$. So $\mathcal{M}[G] \vDash 2^{\aleph_0} \geqslant \kappa = \aleph_\alpha$. Taking $\alpha = 2$ gives us a model of ZFC where the continuum hypothesis fails.

Theorem 16.10. $\mathrm{Con}(\mathsf{ZFC}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

Proof. We won't rewrite this proof, since it follows from the proof last lecture. □

Definition 16.11 (Cohen reals)**.** The g_γ's that we added to $\mathcal{M}$ are called **Cohen reals**.

In particular, we say that $c : \omega \to 2$ is a Cohen real over $\mathcal{M}$ iff

$$\exists H. (H \text{ is } \mathrm{Fn}(\omega, 2)\text{-generic over } \mathcal{M} \text{ and } c = \cup H.)$$

Proposition 16.12 (König)**.** $2^{\aleph_0} \neq \kappa$ *for any* κ *with cofinality* $\aleph_0$.

Proof. This was proven on the first example sheet. □

Moreover, under GCH, for any κ, we have that $\mathrm{cf}(\kappa) \neq \omega$ iff $\kappa^\omega = \kappa$.

In the proof of the power sex axiom in $\mathcal{M}[G]$, we showed that given $a \in \mathcal{M}^{\mathbb{P}}$, a name for its power set was $\mathcal{P}(\mathbb{P} \times \mathrm{ran}(a))$. But this is a very large object, and we want to get a better bound than this.

Theorem 16.13. *Let* $\mathcal{M}$ *be a (countable) transitive model of* ZFC*, assume that* $(\kappa = \aleph_\alpha \wedge \kappa^\omega = \kappa)^{\mathcal{M}}$*. Let* $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ *and let* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$*. Then* $\mathcal{M}[G] \vDash 2^{\aleph_0} = \aleph_\alpha = \kappa$.

Proof. We already have $\mathcal{M}[G] \vDash \mathsf{ZFC} + \kappa = \aleph_\alpha \leqslant 2^{\aleph_0}$.

To show the other direction, let $\dot{x}$ be a name for a subset of ω. For $n \in \omega$, let $E_{\dot{x},n} = \{p \in \mathbb{P} \mid (p \Vdash \check{n} \in \dot{x}) \vee (p \Vdash \check{n} \notin \dot{x})\}$. Then $E_{\dot{x},n}$ is dense in $\mathbb{P}$[2]. Next, for each $n \in \omega$, choose a maximal antichain $A_{\dot{x},n} \subseteq E_{\dot{x},n}$ (this is possible by ES3Q48).

Now, define $\dot{z}_x = \bigcup_{n \in \omega} \{\langle p, \check{n} \rangle \mid p \in A_{\dot{x},n} \wedge p \Vdash \check{n} \in \dot{x}\}$. We call such a name "nice." Then:

Claim: $\mathbb{1} \Vdash x = \dot{z}_x$.

[2]Because if we take any condition q, then either q forces that $\check{n} \in \dot{x}$, or q does not force it. If q does not force it, that means it must be dense below that there is a condition that forces its negation, from the definition.

To show this, it suffices to prove that for every $n \in \omega$, the set $D_{\dot{x},n} = \{q \in E_{\dot{x},n} \mid (q \Vdash \check{n} \in x) \Leftrightarrow (q \Vdash \check{n} \in \dot{z}_x)\}$ is dense. Fix $n \in \omega$ and $p \in \mathbb{P}$. Then since $E_{\dot{x},n}$ is dense, we can fix some condition $p_0 \leqslant p$ such tat $p_0 \in E_{\dot{x},n}$. Now because $A_{\dot{x},n}$ is a maximal antichain, we can fix our condition $q_0 \in A_{\dot{x},n}$ such that $p_0 \,||\, q_0$, i.e. such that p_0 and q_0 are compatible. Now fix $r \leqslant p_0, q_0$. We show that $r \in D_{\dot{x},n}$.

- If $r \Vdash \check{n} \in \dot{x}$, then $q_0 \Vdash \check{n} \in \dot{x}$. Thus $\langle q_0, \check{n}\rangle \in \dot{z}_x$. So $r \Vdash \check{n} \in \dot{z}_x$.
- If $r \Vdash \check{n} \in \dot{z}_x$, then by definition, $\{s \leqslant r \mid \exists \langle q_1, \check{m}\rangle \in \dot{z}_x.\, (s \leqslant q_1 \wedge s \Vdash \check{m} = \check{n})\}$ is dense below r. This can only happen if there is some $\langle q_1, \check{n}\rangle \in \dot{z}_x$ such that $r \,||\, q_1$. So this q_1 must be in the antichain, but also we had that q_0 is in the antichain. However, they are both compatible with r. This is only possible if $q_0 = q_1$, so $\langle q_0, \check{n}\rangle \in \dot{z}_x$. Thus $q_0 \Vdash \check{n} \in x$. So, since $r \leqslant q_0$, $r \Vdash \check{n} \in \dot{x}$. Thus $r \in D_{\dot{x},n}$.

We have shown that every subset of ω has a "nice" name. Computing the number of nice names:

- $|\mathbb{P}| = \kappa$.
- Since $\mathbb{P}$ has the ccc, every antichain is countable.
- Thus, there are at most $(\kappa^\omega \times \omega)^\omega = \kappa$[3] many "nice" names.

So $\mathcal{M}[G] \vDash 2^{\aleph_0} = \kappa$. ☐

16.3 Precise Values of the Continuum

Corollary 16.14. Con(ZFC) $\Rightarrow$ Con(ZFC $+ 2^{\aleph_0} = \aleph_2$). *But becuse we worked in such generality, it could be* $\aleph_{12}$, *it could be* $\aleph_{\omega^3+2}$, *even* $\aleph_{\omega_1}$.

Corollary 16.15. *The following are equiconsistent:*

- ZFC $+ \exists$ *a weakly inaccessible cardinal*
- ZFC $+ \exists$ *a strongly inaccessible cardinal* + GCH.
- ZFC $+ 2^{\aleph_0}$ *is weakly inaccessible.*
- ZFC $+ \exists\kappa$, *weakly inaccessible and not strongly inaccessible.*

Proof. To go from the first to the second, just go into L. This was shown in an example sheet. To go from the third to the fourth, $2^{\aleph_0}$ exhibits such a κ. To go from the fourth to the first is immediate. All that's left is going from the second

[3] κ^ω is the number of antichains, multiplied by ω because our conditions are of the form $\langle p, \check{n}\rangle$, and then all raised to the ω, since we have to do this for each $n \in \omega$.

to the third, "can we have the continuum being weakly inaccessible?" And the answer is yes, we can, just take the forcing $\mathrm{Fn}(\kappa \times \omega, 2)$ where κ is the strongly inaccessible. Then $2^{\aleph_0}$ gives a weakly inaccessible cardinal. $\square$

Remark. When trying to build models of $\mathsf{ZFC} + 2^{\aleph_0} = \kappa$, we often assume GCH just for convenience (i.e. we work in L or something like L) because we usually only care about consistency results anyway.

Example. What happens when we consider $\mathrm{Fn}(\aleph_\omega \times \omega, 2)$. Let G be $\mathrm{Fn}(\aleph_\omega \times \omega, 2)$-generic. Then $\mathcal{M}[G] \models 2^{\aleph_0} \geqslant \aleph_\omega$. Then we want to get an upper bound, which we know (by König's lemma) cannot be $\aleph_\omega$. Assuming GCH,

$$\mathrm{cf}(\kappa) = \omega \Rightarrow \kappa^\omega = \kappa^+.$$

Thus

$$\begin{aligned} \mathcal{M}[G] \models \aleph_\omega &\leqslant 2^{\aleph_\omega} \\ &\leqslant \aleph_\omega^+ \end{aligned}$$

So $\mathcal{M}[G] \models 2^{\aleph_0} = \aleph_{\omega+1}$. So this is what happens in this case.

Another remark on GCH is that we can have $2^{\aleph_0} < \aleph_\omega$, but $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}^{\aleph_0} = \aleph_{\omega+2}$. This is hard to prove and requires the existence of certain large cardinals.

If $\mathcal{M} \models 2^{\aleph_0} = \aleph_\alpha > \aleph_\beta$, and $\mathrm{Fn}(\aleph_\beta \times \omega, 2)$. and G is generic on this set. Then the continuum is greater than the number of reals we add, so we have $\mathcal{M}[G] \models 2^{\aleph_0} = \aleph_\alpha$.

Remark. The following are equiconsistent:

- $\mathsf{ZFC} + \exists$ measurable $+ \mathsf{CH}$
- $\mathsf{ZFC} + \exists$ measurable $+ \neg\mathsf{CH}$.

And the same for I0-I3, and it still holds even if we replace CH with GCH.

Remark. ZFC+ Proper Forcing Axiom implies $2^{\aleph_0} = \aleph_2$.

Lecture 21

17 Generalized Cohen Forcing

Suppose $\mathsf{ZFC} + \mathsf{CH} + 2^{\aleph_1} = \aleph_3$. Our first idea might be to have be our forcing poset $\mathrm{Fn}(\omega_3 \times \omega_1, 2)$. The issue is that this doesn't give us CH.

Proposition 17.1. *Suppose $\mathcal{M}$ is a transitive model of* $\mathsf{ZFC} + \mathsf{GCH}$*, and we have κ so that $(\kappa = \aleph_\alpha \wedge \kappa^\omega = \kappa)^{\mathcal{M}}$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$-generic. Then for any cardinal λ in $\mathcal{M}$ such that $\aleph_0 \leqslant \lambda < \kappa$ in $\mathcal{M}[G]$, then*

$$2^\lambda = \begin{cases} \kappa, & \text{if } \mathrm{cf}(\kappa) > \lambda \\ \kappa^+ & \text{if } \mathrm{cf}(\kappa) \leqslant \lambda. \end{cases}$$

Then there is a natural bijection between $\omega_3 \times \omega$ and $\omega_3 \times \omega_1$. From this, it will follow that in $\mathcal{M}[G]$:

$$2^{\aleph_0} = 2^{\aleph_1} = \aleph_3.$$

Proof. Omitted? or maybe we're about to do it. □

Definition 17.2 (Fn_κ)**.** Let I, J be sets and κ a regular cardinal. Then

- $\mathrm{Fn}_\kappa(I, J) = \{p \mid |p| < \kappa \wedge \mathrm{dom}(p) \subseteq I \wedge \mathrm{ran}(p) \subseteq I\}$.
- Maximal element is $\emptyset$.
- The ordering is given by $q \leqslant p$ iff $p \subseteq q$.

Remark.

- $\mathrm{Fn}_\omega(I, J) = \mathrm{Fn}(I, J)$.
- $\mathrm{Fn}_\kappa(I, J)$ is not absolute for $\kappa > \omega$. Moreover, if $\mathcal{M}$ is a CTM, then $\mathrm{Fn}_\kappa(I, J) \notin \mathcal{M}$, so we need to consider $(\mathrm{Fn}_\kappa(I, J))^{\mathcal{M}}$.
- For $\kappa > \omega$, $\mathrm{Fn}_\kappa(I, J)$ does not have the ccc. (of course, as long as $I, J \neq \emptyset$).
- If G is $\mathrm{Fn}_\kappa(I, J)$-generic over $\mathcal{M}$, then $f = \cup G$ is a function from I to J.

:et $\mathbb{P} = \mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$, and let $\lambda \geqslant \kappa$, κ is regular, and $\lambda^\kappa = \lambda$. By a similar argument to the ω case,

$$f = \cup G, \qquad h_\alpha : \kappa \to 2 \qquad h_\alpha(\beta) = f(\alpha, \beta)$$

This is a sequence of λ-many distinct functions from κ to 2, by the exact same "nice" names argument from earlier, in $\mathcal{M}[G]]$ there are precisely λ-many functions from κ to 2. So $\mathcal{M}[G] \vDash 2^\kappa = \lambda$.

Our aim is to show that $\mathrm{Fn}_\kappa(I, J)$ preserves cardinals.

Definition 17.3 (κ chain condition)**.** This is an analogue to the countable chain condition, we say that $\mathbb{P}$ has the κ **chain condition** iff every antichain has cardinality less than κ

Remark. The ccc is the $\aleph_1$-cc.

Definition 17.4 (Preserving cofinalities $\geqslant \kappa$)**.** We say that $\mathbb{P}$ **preserves cofinalities** $\geqslant \kappa$ iff for every G, $\mathrm{cf}^{\mathcal{M}}(\gamma) = \mathrm{cf}^{\mathcal{M}[G]}(\gamma)$ for all limit ordinals $\gamma \in \mathrm{Ord} \cap \mathcal{M}$ with $\mathrm{cf}^{\mathcal{M}}(\gamma) \geqslant \kappa$.

Lemma 17.5. *Suppose that $\mathbb{P}$ is our forcing poset in $\mathcal{M}$, (κ is a regular cardinal)$^{\mathcal{M}}$, then*

(i) $\mathbb{P}$ preserves cofinalities $\geqslant \kappa$ iff for all $\mathbb{P}$-generic filters G, for all limit ordinals β with $\kappa \leqslant \beta < \mathrm{Ord} \cap \mathcal{M}$:

$$(\beta \text{ is regular})^{\mathcal{M}} \Rightarrow (\beta \text{ is regular})^{\mathcal{M}[G]}.$$

(ii) If $\mathbb{P}$ presrves cofinalities above κ, then $\mathbb{P}$ preserves cardinals above κ.

Proof. Omitted, it might be on the example sheet. □

Lemma 17.6. *Let A, B, $\mathbb{P} \in \mathcal{M}$, ($\kappa$ is regular)$^{\mathcal{M}}$, ($\mathbb{P}$ has the κ-cc), G is $\mathbb{P}$-generic over $\mathcal{M}$. Then for every $f : A \to B$ in $\mathcal{M}[G]$, there is a function $F : A \to \mathcal{P}(B)$ in $\mathcal{M}$ such that for all $a \in A$:*

- $f(a) \in F(A)$
- $(|F(a)| \leqslant \kappa)^{\mathcal{M}}$

Proof. Omitted. □

Theorem 17.7. *If $\mathbb{P} \in \mathcal{M}$, ($\kappa$ is regular)$^{\mathcal{M}}$ and ($\mathbb{P}$ has the κ-cc)$^{\mathcal{M}}$. Then $\mathbb{P}$ preserves cofinalities $\geqslant \kappa$ and hence cardinals $\geqslant \kappa$.*

Proof. Omitted □

From the examples sheet, for any infinite cardinal κ, $\mathrm{Fn}_\kappa(I, J)$ has the $(|J|^{<\kappa})^+$-cc. In particular $\mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$ has the $(2^{<\kappa})^+$-cc. We won't prove this in lecture, it will be on the examples sheet, but we will prove a weaker version that gives us what we need.

Lemma 17.8. *Let κ be a regular cardinal, $(2^{<\kappa} = \kappa)^{\mathcal{M}}$, $(1 \leqslant |J| \leqslant 2^{<\kappa})^{\mathcal{M}}$, then $\mathbb{P} = \mathrm{Fn}_\kappa(I, J)^{\mathcal{M}}$ has the κ^+-cc.*

Proof. If $I = \emptyset$ then trivial, so assume $I \neq \emptyset$. Let W be an antichain in $\mathbb{P}$, To show $|W| \leqslant \kappa$, we construct chains $\langle A_\alpha \,|\, \alpha < \kappa \rangle$ in I, and $\langle W_\alpha \,|\, \alpha < \kappa \rangle$ such that:

(i) For $\alpha < \beta < \kappa$, $A_\alpha \subseteq A_\beta \subseteq I$, $W_\alpha \subseteq W_\beta \subseteq W$.

(ii) For γ a limit, $A_\gamma = \cup_{a<\gamma} A_\alpha$, $W_\gamma = \cup_{\alpha<\gamma} W_\alpha$.

(iii) $W = \cup_{\alpha<\kappa} W_\alpha$

(iv) For all $\alpha < \kappa$, $|W_\alpha|$, $|A_\alpha| \leqslant \kappa$.

Assuming we can do this, by the regularity of κ^+, $|W| \leqslant \kappa$.

How do we build this? Well, we'll start by setting $A_0 = W_0 = \emptyset$. Then assume that A_α, W_α are defined. Then for each $p \in \mathbb{P}$ with $\mathrm{dom}(p) \subseteq A_\alpha$, using AC, choose $q_p \in W$ such that $p = q_p \restriction A$ (if it exists, if it doesn't we just move on and everything is fine).

Then $W_{\alpha+1} = W_\alpha \cup \{q_p \,|\, \mathrm{dom}(p) \subseteq A_\alpha\}$. If $\mathrm{dom}(p)$ were contained in A_β, with $\beta < \alpha$, then the q_p we pick here should be the same as the q_p that was picked at stage β.

Then define $A_{\alpha+1} = \cup\{\mathrm{dom}(q) \,|\, q \in W_{\alpha+1}\}$.

Finally, set $A_\gamma = \cup_{\alpha<\gamma} A_\alpha$, $W_\gamma = \cup_{\alpha<\gamma} W_\alpha$ when γ is a limit, and then define $A = \cup_{\alpha<\kappa} A_\alpha$.

Claim: $W = \cup_{\alpha<\kappa} W_\alpha$.

By construction, $\cup_{\alpha<\kappa} W_\alpha \subseteq W$. Fix $q \in W$. Firstly, $\mathrm{dom}(q) \cap A \neq \emptyset$. Otherwise, take $q_1 \in W_1$. Then $\mathrm{dom}(q_1) \subseteq A$. So if $\mathrm{dom}(q_1) \cap \mathrm{dom}(q) = \emptyset$, then $\mathrm{dom}(q_1) \,||\, \mathrm{dom}(q)$, contradicting $q_1, q \in W$.

Since $\mathrm{dom}(q) \cap A = \emptyset$ and $|\,\mathrm{dom}(q)| < \kappa$, $\mathrm{dom}(q) \cap A = \mathrm{dom}(q) \cap A_\alpha$ for some α.

Now define $p = q \restriction A_\alpha$. Then there is some $q' \in W_{\alpha+1}$ such that $q' \restriction A_\alpha = p$. Since $\mathrm{dom}(q') \subseteq A$, so $q \,||\, q'$. Since W is an antichain, this is only possible if $q = q'$.

Claim: For all $\alpha < \kappa$, $|W_\alpha|$, $|A_\alpha| \leqslant \kappa$. Proven by induction.

- The limit cases follow by regularity.
- If $|W_\alpha| \leqslant \kappa$, then $|A_\alpha| \leqslant \kappa$ because q lie in $\mathrm{Fn}_\kappa(I, J)$, the κ is the important part here.
- Suppose $|W_\alpha| \leqslant \kappa$. Then since every q that is added to W_α is chosen from some $p \in \mathbb{P}$, $\mathrm{dom}(p) \subseteq A_\alpha$, so

$$|W_{\alpha+1}| \leqslant |W_\alpha| + |\{p \in \mathbb{P} \mid \mathrm{dom}(p) \subseteq A_\alpha\}|$$

- Since $|A_\alpha| \leqslant \kappa$, and $|\,\mathrm{dom}(p)| < \kappa$ $|[A_\alpha]^{<\kappa}| \leqslant \kappa^{<\kappa} = 2^{<\kappa} = \kappa$.

So we have $\mathbb{P} = \mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$, $\mathcal{M}[G] \vDash 2^\kappa = \lambda$ and all cardinals $\geqslant \kappa^+$ are preserved.

Lecture 22

17.1 Closure and Distributivity

Definition 17.9 (κ-closed)**.** A poset $\mathbb{P}$ is **κ-closed** iff $\forall \delta < \kappa$, every decreasing sequence $\langle p_\alpha \mid \alpha < \delta \rangle$ in $\mathbb{P}$ has a lower bound.

I.e. for all $\alpha < \beta < \delta$, if $p_\beta < p_\alpha$ then $\exists q, \forall \alpha < \delta .\, q \leqslant p_\alpha$

Definition 17.10 ($< \kappa$-closed)**.** $\mathbb{P}$ is $< \kappa$-closed iff it is λ-closed for all $\lambda < \kappa$ cardinals. That is, any decreasing sequence of conditions $\langle p_\alpha \,|\alpha < \gamma \rangle$ has a lower bound.

Definition 17.11 ($< \kappa$-distributive)**.** $\mathbb{P}$ is **$< \kappa$-distributive** iff for every $\gamma < \kappa$, the intersection of $< \kappa$-many open dense sets is open and dense. Remember that:

- D is dense if $\forall p in \mathbb{P} \exists q \in D .\, q \leqslant p$.
- D is open if $\forall p \in D, \forall q \in \mathbb{P} .\, (q \leqslant p \to q \in D)$.

Lemma 17.12. *$\mathbb{P}$ is $< \kappa$-closed $\Rightarrow$ $\mathbb{P}$ is $< \kappa$-distributive.*

Proof. Omitted □

Lemma 17.13 (I)**.** *f (κ is regular)$^{\mathcal{M}}$ then $\mathrm{Fn}_\kappa(I, J)^{\mathcal{M}}$ is $< \kappa$-closed.*

Proof. Omitted □

Theorem 17.14. *Let $A, B, \mathbb{P} \in \mathcal{M}$, ($\kappa$ is a cardinal)$^{\mathcal{M}}$, $(|A| < \kappa)^{\mathcal{M}}$, ($\mathbb{P}$ is κ-distributive)$^{\mathcal{M}}$. Let G be generic, then any function $f \in \mathcal{M}[G]$ with $f : A \to B$ has $f \in \mathcal{M}$.*

Proof. Suffices to prove the statement for $A = \delta$ when $\delta < \kappa$. Suppose $\mathcal{M}[G] \vDash f : \delta \to \beta$. By the forcing theorem, fix $p \in G$ such that $p \Vdash f : \check{\delta} \to \check{\beta}$. $\dot{f}$ is a name for f.

For each $\alpha < \delta$, let $D_\alpha = \{q \leqslant p \,|\, \exists x \in B.\, q \Vdash \dot{f}(\check{\alpha}) = \check{x}\}$. This is open and dense, so since $\mathbb{P}$ is κ-distributive, we can take $D = \cap_{\alpha<\kappa} D_\alpha$ is open and dense below p (open is a technical condition which makes things work, the key point is that it is dense). So fix $q \in D \cap G$. To argue this is in $\mathcal{M}$, for each $\alpha < \delta$, choose $x_\alpha \in \beta$ such that $q \Vdash f(\check{\alpha}) = \dot{x}_\alpha$. Now define

$$\begin{aligned} g : \delta &\to \beta \\ \alpha &\mapsto x_\alpha \end{aligned}$$

Then $g \in \mathcal{M}$. But for any $\alpha < \delta$, we have

$$q \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha = \check{g}(\check{\alpha})$$

. So $\mathcal{M}[G] \vDash f = g$. So $f \in \mathcal{M}$. □

Theorem 17.15. *Let $I, J, \kappa \in \mathcal{M}$, suppose ($\kappa$ is regular)$^{\mathcal{M}}$, $(2^{<\kappa} = \kappa)^{\mathcal{M}}$, and $(|J| \leqslant \kappa)^{\mathcal{M}}$. Then $\mathrm{Fn}_\kappa(I, J)^{\mathcal{M}}$ preserves cofinalities and hence cardinals.*

Proof. Recall, it suffices to show that for every limit ordinal $\beta < \mathrm{Ord} \cap \mathcal{M}$, if β is regular in $\mathcal{M}$, then β is regular in $\mathcal{M}[G]$. There are two cases to consider:

($\beta > \kappa$) Since $(|J| \leqslant \kappa = 2^{<\kappa})^{calM}$., $\mathrm{Fn}_\kappa(I, J)^{\mathcal{M}}$ has the κ^+-cc. So $\mathrm{Fn}_\kappa(I, J)$ oreserves all cardinals and cofinalities above κ^+. In particular, if (β is regular)$^{\mathcal{M}}$ then (β is regular)$^{\mathcal{M}[G]}$.

($\beta \leqslant \kappa$) We will show that if (β is singular)$^{\mathcal{M}[G]}$, then (β is singular)$^{\mathcal{M}}$. In $\mathcal{M}[G]$, fix $\delta < \beta$, $f@\delta \to \beta$ cofinal. Since $\mathcal{M}$ is transitive, $\delta \in \mathcal{M}$. Since $\mathbb{P}$ is κ-closed, $f \in \mathcal{M}$. So (β is singular)$^{\mathcal{M}}$. Then we are done by the contrapositive.

□

Theorem 17.16. *Suppose* (κ, λ *are cardinals*)$^{\mathcal{M}}$*, that* $\aleph_0 \leqslant \kappa \leqslant \lambda$*, and that* ($\kappa$ *is regular*)$^{\mathcal{M}}$*,* $(2^{<\kappa} = \kappa)^{\mathcal{M}}$*,* $(\lambda^{\kappa} = \lambda)^{\mathcal{M}}$*. Let* $\mathbb{P} = \mathrm{Fn}_{\kappa}(\lambda \times \kappa, 2)$*,* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$*. Then* $\mathbb{P}$ *preserves cardinals and* $\mathcal{M}[G] \vDash 2^{\kappa} = \lambda$*.*

Proof. Comes from the above □

17.2 Fixing multiple values of the continuum

Theorem 17.17. *Suppose there is a countable transitive model of* $\mathsf{ZFC} + \mathsf{GCH}$*, then there is a countable transitive model of* ZFC *satisfying any of the following statements:*

(i) $\mathsf{CH} + 2^{\aleph_1} = \aleph_3$

(ii) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_5 + 2^{\aleph_2} = \aleph_{\omega+5}$.

(iii) For a fixed $n \in \omega$*, for all* $m \leqslant n$*,* $2^{\aleph_m} = \aleph_{2m+3}$.

Proof. Let $\mathcal{M}$ be a ctm of $\mathsf{ZFC} + \mathsf{GCH}$. Then

(i) Let $\mathbb{P} = \mathrm{Fn}_{\aleph_1}(\omega_3 \times \omega_1, 2)$. If G is $\mathbb{P}$-generic, then

$$\mathcal{M}[G] \vDash 2^{\aleph_1} = \aleph_3.$$

Also, as $\mathbb{P}$ is ω_1-closed, it doesn't add new functions from ω to 2, so $(\mathsf{CH})^{\mathcal{M}[G]}$.

(ii) First, force $\mathbb{P}_0 = \mathrm{Fn}_{\aleph_2}(\omega_{\omega+5} \times \omega_2, 2)^{\mathcal{M}}$. Let G_0 be $\mathbb{P}_0$-generic. By closure, $(2^{<\aleph_1} = \aleph_1)^{\mathcal{M}[G_0]}$, $(\aleph_5^{\aleph_1} = \aleph_5)^{\mathcal{M}[G_0]}$. Now let $\mathbb{P}_1 = \mathrm{Fn}_{\aleph_0}(\omega_5 \times \omega, 2)^{\mathcal{M}[G_0]}$. Let G_1 be $\mathbb{P}_1$-generic, then $\mathcal{M}[G_1] \vDash 2^{\aleph_0} = 2^{\aleph_1} = \aleph_5$ as well as $2^{\aleph_2} = \aleph_{\omega+5}$. Recall a result that if we make the continuum have size (e.g.) $aleph_5$, and we are beginning from a model of GCH then for any cardinal $\aleph_0 \leqslant \lambda < \kappa$ in $\mathcal{M}$, in $\mathcal{M}[G]$ we have $2^{\lambda} = \kappa$ or $2^{\lambda} = \kappa^+$ depending on $\mathrm{cf}(\kappa)$.

(iii) Similar.

□

Remark.

- It is necessary to start at the largest cardinals and work backwards for cardinal arithmetic to work.
- Our iterative approach works for any *finite* number of cardinals.
- See later how to get forcings of the sort $2^{\aleph_n} = 2^{\aleph_{2n+3}}$ for all $n \in \omega$, i.e. infinitary forcings.

Proposition 17.18. *Suppose $\mathcal{M}$ is a countable transitive model of* ZFC*, $\mathcal{M} \models 2^{\aleph_0} = \aleph_\alpha$ for $\alpha \geqslant 1$. Let $\mathbb{P} = \mathrm{Fn}_{\aleph_1}(\kappa \times \aleph_1, 2)$. Then if we let G be $\mathbb{P}$-generic, in $\mathcal{M}[G]$ we get a model of* CH*, and all cardinals δ of $\mathcal{M}$ with $(\aleph_1 \leqslant \alpha \leqslant \aleph_\alpha)^{\mathcal{M}}$.*

In particular $\aleph_\alpha^{\mathcal{M}} \neq \aleph_\alpha^{\mathcal{M}[G]}$. This is because we can encode subsets of $\aleph_0$ into the forcing which we are using here.

Proof. Example sheet 4 (I think).

Lecture 23 **Observation:** If $(\delta$is a cardinal$)^{\mathcal{M}}$, and $(\delta > |\mathbb{P}|)^{\mathcal{M}}$, then $(\delta$is a cardinal$)^{\mathcal{M}[G]}$. Observe also that $\mathbb{P}$ has the $|\mathbb{P}|^+$-cc.

Recall that $p \Vdash \exists x \varphi(x)$ iff $\{q \leqslant p \mid \exists \dot{x} \in V^{\mathbb{P}}. q \Vdash \varphi(x)\}$ is dense below p. In most cases, the witness $\dot{x}$ doesn't depend on the filter. For example, $p \Vdash \exists x(\dot{a} \in x \wedge \dot{b} \in x)$. We don't need the generic to find a name as $\dot{x} = |(\rangle\langle(| \, \dot{a}, \dot{b})$.

Lemma 17.19 (The mixing lemma, ZFC)**.** *Suppose $(p \Vdash \exists x \varphi(x))^{\mathcal{M}}$, then $\exists \dot{x} \in \mathcal{M}^{\mathbb{P}}$ such that $(p \Vdash \varphi(\dot{x}))^{\mathcal{M}}$.*

Proof. Since the set $\{q \leqslant p \mid \exists x \in \mathcal{M}^{\mathbb{P}}. q \Vdash \varphi(x)\}$ is dense below p, it contains a maximal antichain, D (this requires choice). Now, for each of our conditions $q \in D$, choose $\dot{x}_q \in \mathcal{M}^{\mathbb{P}}$ such that $q \Vdash \varphi(\dot{x}_q)$. Without loss of generality, we may assume that if $\langle r, \dot{y} \rangle \in \dot{x}_q$, then $r \leqslant q$. This is because if $r \nleq q$ then there are two cases:

- $r \perp q$, but if this is the case, then $q \Vdash \dot{x}_q = \dot{x}_q \setminus \langle r, \dot{y} \rangle$.
- If $r || q$, then define $\dot{x}'_q = (\dot{x}_q \setminus \langle r, \dot{y} \rangle) \cup \{\langle s, \dot{y} \rangle \mid s \leqslant r, q\}$. Again, $q \Vdash \dot{x}_q = \dot{x}'_q$.

Now if $q, q' \in D$ and $q \neq q'$, then $q \perp q'$ so $q' \Vdash \dot{x}_q = \emptyset$. Next, let $\dot{x} = \cup\{\dot{x}_q \mid q \in D\}$. Then if $q \in D$, $q \Vdash \dot{x} = \dot{x}_q$. By the Forcing theorem, $q \Vdash \varphi(\dot{x})$ (since it forces it for $\dot{x}_q$, and that $\dot{x} = \dot{x}_q$).

It remains to prove that $p \Vdash \varphi(\dot{x})$. Suppose not, then we can fix an $r \leqslant p$ such that $r \Vdash \neg\varphi(\dot{x})$. Since D is a *maximal* antichain, we can fix $q \in D$ such that $q || r$. But now if we take $s \leqslant q, r$, we have $s \Vdash \varphi(\dot{x})$, since it is below q, and $s \Vdash \neg\varphi(\dot{x})$, since it is below r. Contradiction, so $p \Vdash \varphi(\dot{x})$. ☐

Question: Given cardinals $\kappa \leqslant \lambda$ in $\mathcal{M}$, can we find a generic G (for some $\mathbb{P}$) such that $\mathcal{M}[G] \models \lambda = \kappa^+$.

First, observe that in order to find such a generic, then $(\lambda$ must be regular$)^{\mathcal{M}}$. If

$f : \alpha \to \lambda$ cofinal, $\alpha < \lambda$, $f \in \mathcal{M}$. Then $f \in \mathcal{M}[G]$, so $\mathrm{cf}^{\mathcal{M}[G]}(\lambda) \leqslant \mathrm{cf}^{\mathcal{M}[G]}(\alpha) \leqslant |\alpha|^{\mathcal{M}[G]} < \lambda$. However, this is the only restriction.

Theorem 17.20. *Let (κ be regular)$^{\mathcal{M}}$, ($\delta > \kappa$), (δ is a cardinal)$^{\mathcal{M}}$. Let $(\lambda = \delta^+)^{\mathcal{M}}$. Let G be $\mathrm{Fn}_\kappa(\kappa, \delta)$-generic over $\mathcal{M}$. Then in $\mathcal{M}[G]$:*

- $|\delta| = \kappa$.
- *Every cardinal $\alpha \leqslant \kappa$ in $\mathcal{M}$ remains a cardinal.*
- *If $\delta^{<\kappa} = \delta$, then every cardinal α such that $\alpha > \delta$ in $\mathcal{M}$ remains a cardinal in $\mathcal{M}[G]$.*

In particular, if $\delta^{<\kappa} = \delta$, then $\mathcal{M}[G] \vDash \lambda = \kappa^+$.

Proof. $\cup G : \kappa \to \delta$ is a surjection, so $|delta| = |\kappa|$. So there are no cardinals between δ and λ.

Since κ is regular, $\mathrm{Fn}_\kappa(\kappa, \delta)$ is $< \kappa$-closed, so every cardinal $\alpha \leqslant \kappa$ is preserved.

And finally, if $\delta^{<\kappa} = \delta$, then $|\mathrm{Fn}_\kappa(\kappa, \delta)| = \delta$. So $\mathrm{Fn}_\kappa(\kappa, \delta)$ has the δ^+-cc. (i.e. λ-cc). Thus every cardinal $\alpha > \delta$ (in particular, λ) is preserved. □

The other case is that λ is weakly inaccessible. Here we use a forcing called the Lévy collapse.

Definition 17.21 ($\mathrm{Col}(\kappa, < \lambda)$)**.** Let $\lambda > \kappa$ be infinite ordinals, and let $\mathrm{Col}(\kappa, < \lambda)$ consist of all functions p such that:

- p is a partial function from $\kappa \times \lambda$ to λ.
- $|\mathrm{dom}(p)| < \kappa$.
- $p(\alpha, \beta) < \beta$ for each $(\alpha, \beta) \in \mathrm{dom}(p)$.

And we say $q \leqslant p$ iff q extends p as a function

Theorem 17.22. *Let κ be regular, and suppose $\lambda > \kappa$ is weakly inaccessible. Let G be $\mathrm{Col}(\kappa, < \lambda)$-generic over $\mathcal{M}$. Then in $\mathcal{M}[G]$:*

- *Every ordinal β with $\kappa \leqslant \beta < \lambda$ has cardinality κ.*
- *Every cardinal $\leqslant \kappa$ and $\geqslant \lambda$ remains a cardinal.*

Hence $\mathcal{M}[G] \vDash \lambda = \kappa^+$.

Proof. For each $\beta < \lambda$, define $G_\beta : \kappa \to \beta$ by $G_\beta(\alpha) = G(\alpha, \beta)$. So if $\kappa \leqslant \beta < \lambda$, $\mathcal{M}[G] \vDash |\beta| = |\kappa|$. Next $\mathrm{Col}(\kappa, < \lambda)$ is $<$ *kappa*-closed, so it preserves cardinals $\leqslant \kappa$. Finally ,$|\,\mathrm{Col}(\kappa, < \lambda)| = \lambda$, so $\mathrm{Col}(\kappa, < \lambda)$ has the λ-cc, and preserves cardinals $\geqslant \lambda$[4] □

Remark. λ is weakly compact iff it is inaccessible and satisfies something called the **tree property**.

Claim: If G is $\mathrm{Col}(\aleph_0, < \lambda)$-generic, then we have that $\mathcal{M}[G] \vDash \aleph_1$ has the tree property.

Remark. This shows that λ being a limit cardinal is not absolute between transitive models $\mathcal{M}$ and $\mathcal{N}$, *even if* λ being a cardinal is!

The final question we want to address in this course is: Is the following consistent:

$$\mathsf{ZFC} + \forall n \in \omega.\, 2^{\aleph_n} = \aleph_{2n+3}.$$

Remark. We have the consistency of $\mathsf{ZFC} + \forall n \leqslant k.\, 2^{\aleph_n} = 2^{\aleph_{2n+3}}$, for any $k \in \omega$, because we can do arbitrarily finite forcing, but we can't yet do infinite forcings.

The answer to this question (spoiler) is yes! We use Easton's forcing. Then our follow-up is: What are the restrictions on the function $F : \mathrm{Card} \to \mathrm{Card}$, where $F(\aleph_\alpha) = 2^{\aleph_\alpha}$?

Lecture 24

Theorem 17.23. *Let κ be regular, $\lambda > \kappa$ is strongly inaccessible, G a $\mathrm{Col}(\kappa, < \lambda)$-generic. Then $\mathcal{M}[G] \vDash \lambda = \kappa^+$.*

Remark. Suppose λ were weakly inaccessible, and $2^{\aleph_0} > \lambda$, then $\mathrm{Col}(\aleph_1, < \lambda)$ has an antichain of length $2^{\aleph_0}$.

Proof. For $A \subseteq \omega$, define $p_A : \{w\} \times [\omega, \omega + \omega)$ by:

$$p_A(\alpha, \omega + n) = \begin{cases} 0, & n \in A \\ 1, & n \notin A \end{cases}$$

Then if $A \neq B$, then $p_A \perp p_B$. From last time, we know that all cardinals δ with $\kappa < \delta < \lambda$ are collapsed. We need to prove that λ is a cardinal, which follows from the λ-cc. Given $p \in \mathrm{Col}(\kappa, < \lambda)$, let $^{(}p) = \{\beta : \exists \alpha.\, (\alpha, \beta) \in \mathrm{dom}(p)\}$. So

[4] Actually, we only have λ^+-cc right now, but there is an argument which shows it preserves cardinals $\geqslant \lambda$.

$|^{(}p)| < \kappa$.

Let W be an antichain. Consider chains $\langle A_\alpha \,|\, \alpha < \kappa\rangle$ and $\langle W_\alpha \,|\, \alpha < \kappa\rangle$ such that:

1. for $\alpha < \beta < \kappa$, $A_\alpha \subseteq A_\beta$, $W_\alpha \subseteq W_\beta$.
2. If γ is a limit $\cup_\alpha A_\alpha = A_\gamma$, $\cup_{\alpha<\gamma} W_\alpha = W_\gamma$.
3. $W = \cup_{\alpha<\kappa} W_\alpha$
4. For all $\alpha < \kappa$, $|A_\alpha|, |W_\alpha| < \lambda$.

Assuming this, by regularity of λ, $|W| = |\cup_{\alpha<\kappa} W_\alpha| < \lambda$.

To find these sets, set $A_0, W_0 = \emptyset$. Assume A_α, B_α. Then for $p \in \mathrm{Col}(\kappa, <\lambda)$ with $^{(}p) \subseteq A_\alpha$, using the axiom of choice, choose $q_p \in W$ such that $p = q_p \restriction (\kappa \times^{(} p))$ (if it exists).

Then $W_{\alpha+1} = \{q_p \,|\, ^{(}p) \subseteq A_\alpha\}$. And $A_{\alpha+1} = \cup\{^{(}q) \,|\, q \in W_\alpha\}$.

Claim: $W = \cup_{\alpha<\kappa} W_\alpha$.

The proof of this is the same as the proof in the $\mathrm{Fn}_\kappa(I, J)$ proof.

Claim: For $\alpha < \kappa$, $|W_\alpha|, |A_\alpha| < \lambda$.

We prove this by induction on α.

- The limit cases follow by regularity.
- If $|W_{\alpha+1}| < \lambda$, then $|A_{\alpha+1}| \leqslant \kappa \cdot \lambda = \lambda$, since it is a union of less than lambda many things each of size less than κ.
- If $|W_\alpha| < \lambda$, then since every q that is added to W_α is chosen from some $p \in \mathrm{Col}(\kappa, <\lambda)$ with $^{(}p) \subseteq A_\alpha$. So $|W_{\alpha+1}| \leqslant |A_\alpha|^{<\kappa}$. Since λ is strongly inaccessible, this must be less than λ.

□

Corollary 17.24. *If* $\mathrm{Con}(\mathsf{ZFC} + \exists$*inacc. cardinal.*), *then* $\mathrm{Con}(\mathsf{ZFC} + \aleph_1^V$ *is inaccessible in* L).

Proof. If we start with $V = L$, and let G be $\mathrm{Col}(\omega_1, <\lambda)$-generic. Then $\mathcal{M}[G] \models \lambda = \aleph_1$, so $\mathcal{M}[G] \models (\lambda \text{ is inacc.})^L$. □

Remark. If $V \models \mathsf{ZFC} + \kappa$ is measurable, then $\aleph_1^V$ is inaccessible in L.

17.3 Product Forcing

Definition 17.25 (Product Order)**.** Suppose we have two forcing notions $(\mathbb{P}, \leqslant_{\mathbb{P}}$), and $(\mathbb{Q}, \leqslant_{\mathbb{Q}})$, then we can define the **product order**: $\leqslant$ on $\mathbb{P} \times \mathbb{Q}$, by

$$\langle p_1, q_1 \rangle \leqslant \langle p_2, q_2 \rangle \text{ iff } p_1 \leqslant_{\mathbb{P}} p_2 \text{ and } q_1 \leqslant_{\mathbb{Q}} q_2$$

Definition 17.26 (Projections)**.** Given a $\mathbb{P} \times \mathbb{Q}$-generic filter, G, over $\mathcal{M}$. Let $G_0 = \{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}. (p, q) \in G\}$, and $G_1 = \{q \in \mathbb{Q} \mid \exists p \in \mathbb{P}. (p, q) \in G\}$. These are called the **projections** of G.

Lemma 17.27. *Suppose $\mathcal{M}$ is a transitive model of* ZFC*, and $\mathbb{P}, \mathbb{Q} \in \mathcal{M}$. Let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then TFAE:*

1. *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over $\mathcal{M}$.*
2. *G is $\mathbb{P}$-generic over $\mathcal{M}$, and H is $\mathbb{Q}$-generic over $\mathcal{M}[G]$.*
3. *H is $\mathbb{Q}$-generic over $\mathcal{M}$, and G is $\mathbb{P}$-generic over $\mathcal{M}[H]$.*

Moreover, if (i) holds, then $\mathcal{M}[G \times H] = \mathcal{M}[G][H] = \mathcal{M}[H][G]$.

Proof. The first part is on example sheet 4. The idea is to prove first that they are filters, and then prove genericity, for both (i) implies (ii) and (ii) implies (i). To prove the last claim:

Recall the last part of the Generic Model Theorem which said that if $\mathcal{M}$ is transitive, $\mathcal{M} \vDash \mathsf{ZF}$, $\mathbb{P} \in \mathcal{M}$, $G \subseteq \mathbb{P}$, $\mathbb{1} \in G$. Then if $\mathcal{N}$ is a transitive model of ZF, $\mathcal{M} \subseteq \mathcal{N}$ is a definable class in $\mathcal{N}$, then $\mathcal{M}[G] \subseteq \mathcal{N}$.

Since $M \subseteq \mathcal{M}[G][H]$, and $G \times H \in \mathcal{M}[G][H]$. Then $\mathcal{M}[G \times H] \subseteq \mathcal{M}[G][H]$.

For the other direction, $G \in \mathcal{M}[G \times H]$, and $\mathcal{M} \subseteq \mathcal{M}[G \times H]$, so $\mathcal{M}[G] \subseteq \mathcal{M}[G \times H]$, and then since also $H \in \mathcal{M}[G \times H]$, $\mathcal{M}[G][H] \subseteq \mathcal{M}[G \times H]$. $\square$

18 Easton Forcing

** Proof Non-examinable **.

Return to the model of $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_5$. We started with $\mathcal{M} \vDash \mathsf{ZFC} + \mathsf{GCH}$, $\mathrm{Fn}(\omega_3 \times \omega, 2)^{\mathcal{M}}$, G_0 generic over this. And $\mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2)^{\mathcal{M}[G_0]}$.

Then $\mathcal{M}[G_0][G_1] \vDash \mathsf{CH}$. So we did it the other way around:

We took $\mathbb{P}_0 = \mathrm{Fn}_{\omega_1}(\omega_5 \times \omega_1, 2)^{\mathcal{M}}$, and had G_0 generic over that. Then we took $\mathbb{P}_1 = \mathrm{Fn}(\omega_3 \times \omega, 2)^{\mathcal{M}G_0}$, and had G_1 generic over that. Then $\mathcal{M}[G_0][G_1]$ gives us the model we want.

However, $\mathbb{P}_0$ is $< \omega_1$ closed, so it doesn't add new sequences of length ω. Thus $\mathbb{P}_1 = \mathrm{Fn}^{\mathcal{M}}_{\omega_3 \times \omega, 2}$, the smaller forcing isn't changed in $\mathcal{M}[G_0]$. So we can define the forcing $\mathbb{P}_0 \times \mathbb{P}_1$ over $\mathcal{M}$, and $G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over $\mathcal{M}$.

This leads to an obvious candidate for $2^{\aleph_n} = \aleph_{2n+3}$:

$$\mathbb{P} = \prod_{n<\omega} \mathrm{Fn}_{\omega_n}(\omega_{2n+3} \times \omega_n, 2).$$

It turns out that this works.

Theorem 18.1 (Easton (Set version))**.** *Let $\mathcal{M}$ be a countable transitive model of* $\mathsf{ZFC} + \mathsf{GCH}$*. Let S be a set of regular cardinals in $\mathcal{M}$, and let $F : S \to \mathrm{Card} \cap \mathcal{M}$ be a function in $\mathcal{M}$ such that for all $\kappa \leqslant \lambda$ in S, the following hold:*

(i) $F(\kappa) > \kappa$*. (restriction comes from Cantor)*

(ii) $F(\kappa) \leqslant F(\lambda)$*. (restriction comes from montonicity)*

(iii) $\mathrm{cf}(F(\kappa)) > \kappa$*. (restriction comes from König)*

Then there is a generic extension $\mathcal{M}[G]$ of $\mathcal{M}$ such that $\mathcal{M}$ and $\mathcal{M}[G]$ have the same cardinals, and for all $\kappa \in S$, $\mathcal{M} \vDash 2^\kappa = F(\kappa)$.

Remark. This can be generalised by a virtually identical proof to the class of regular cardinals. This, however, needs a "class-forcing", so $\mathbb{P}$ needs to be a proper class. The big issue with class forcings is that when $\mathbb{P}$ is a proper class, it is nontrivial to show $\mathcal{M}[G] \vDash \mathsf{ZFC}$, and in fact it sometimes does not model ZFC. For example $\mathrm{Fn}(\mathrm{Ord} \times \omega, 2)$ makes $2^{\aleph_0}$ a proper class. Or $\mathrm{Fn}(\omega, \mathrm{Ord})$ which adds a surjection from ω to Ord.

In fact, $\Vdash$ may not be defined.