# Infinite Games in Set Theory: Large Cardinals from Determinacy

## Contents

# Introduction

In this essay, we prove a result of Harrington that analytic determinacy implies the existence of $0^\#$ (*zero sharp*), which is a particular type of large cardinal hypothesis. To provide the necessary background for this proof, we will first outline some basic concepts and results in infinite games. Then in section two we will define analytic determinacy and provide the necessary context of Borel pointclasses, projective pointclasses, and the boldface and lightface hierarchies. We will also (very) briefly touch on some historical determinacy results. Section three will then serve to elucidate $0^\#$. The presentation we will give will demonstrate how $0^\#$ arose historically, as a construction flowing naturally from ideas in model theory, although towards the end of the section we will touch on a more contemporary perspective of $0^\#$, which is how we will use it in the rest of the essay. Section four will outline a proof of Harrington's theorem: that analytic determinacy implies the existence of $0^\#$. The beginning of this proof will provide a proof of the existence of $0^\#$ from a statement we will call Harrington's principle. The remainder of section four will prove that analytic determinacy implies this principle and thus implies the existence of $0^\#$. Finally, we will briefly discuss the consistency strength of analytic determinacy and $0^\#$, as well as some related results, such as the Martin-Harrington theorem.

# 1 Infinite Games

To begin, we cover some basic definitions of the topic of infinite games as introduced by Gale and Stewart [4]. We will also summarise the basics of determinacy. We will see that finite games are always determined, and that there is a set which is *not* determined. This will put us on good footing before we delve deeper into determinacy in the next section.

## 1.1 Basic Definitions

**Definition 1.1** (Infinite game)**.** An **infinite game** (or **Gale-Stewart game**) is a game with five basic properties:

(i) There are two players, who alternately take turns.

(ii) The game can either be won by player I or won by player II, not both. There are no draws.

(iii) The game is played for $\omega$ steps.

(iv) The players of the game have perfect information about the moves the other player has made, and the state of the game.

(v) The players are able to recall all moves that have been played in the game thus far (they have *perfect recall*).

The possible moves that a player can make are contained in some set. For the purposes of this essay, the players will always make moves in $\omega$. A complete run of a game (the ordered collection of all played moves) is an element of ${}^{\omega}\omega$, the set of infinite sequences of natural numbers.

**Definition 1.2** (Payoff set)**.** We define the **payoff set** of a game to be a set $A \subset {}^{\omega}\omega$ such that if the complete run of a game $x \in A$, then player I wins. Otherwise, player II wins. A game with payoff set $A$ is denoted $G(A)$.

**Definition 1.3** (Strategy)**.** A **strategy** for a player is a function $f : {}^{<\omega}\omega \to \omega$. This assigns to each finite sequence of elements of $\omega$ a "response" in $\omega$. We only care about the map from odd-length sequences for player II and from even-length sequences for player I. We call a strategy a *winning strategy* if it beats anything the other player could play against it.

## 1.2 Determinacy

For finite games, work by Zermelo, (and generalised by König and Kalmár [10, §1]) showed:

**Theorem 1.4** (Zermelo's Theorem)**.** *For any finite, two-player game of perfect information without draws, either*

*(i) Player* I *has a winning strategy, or*

*(ii) Player* II *has a winning strategy.*

*Proof.* Recurse on the end-states of the game. See [16, Appendix] for a full account. ☐

So, with perfect play by both players, every finite game has a winner. This leads to a natural definition in the context of infinite games.

**Definition 1.5** (Determined)**.** A set $A \subset {}^{\omega}\omega$ is **determined** if, in $G(A)$, one of the players has a winning strategy.

**Example 1.6.** A basic example of a determined payoff set is the set

$$A = \{f \in {}^{\omega}\omega \,:\, \forall n \in \omega\, (f(n) \leqslant 10)\}$$

This corresponds to the game where player I wins iff no player plays a number larger than 10. Player II has an easy winning strategy for this game, which is to play 11 on the first move.

Another example of a determined payoff set is given by

$$A = \{f \in {}^{\omega}\omega \,:\, \forall y \in \omega\, \exists n \in \omega\, (f(n) = y)\}.$$

Player I has a winning strategy for this game which consists of first playing 0, then playing 1, and in general, playing $n - 1$ on her $n^{\text{th}}$ move.

So, is anything *not* determined? We will be working in ZFC, so the answer to this question is yes. However, it is not obvious that this is the case. That there is a non-determined game in ZFC relies on the axiom of choice, and in fact it *is* consistent with ZF that every game with players making moves in $\omega$ is determined.

**Lemma 1.7.** *There is a non-principal ultrafilter on $\omega$.*

This is well-known, so we won't prove it in detail. It can be shown by first taking the collection of cofinite sets on $\omega$; then by applying Zorn's lemma[1] extend this collection of sets to an ultrafilter. This gives an ultrafilter $U$ on $\omega$ which is non-principal.

**Theorem 1.8.** *There is a non-determined infinite game.*

*Proof.* Suppose there is a game where player I and player II take turns playing increasing elements of $\omega$. Then for a game with the following moves

$$\begin{array}{r|ccccc} \text{I} & x_0 & & x_1 & & \cdots \\ \text{II} & & y_0 & & y_1 & \cdots \end{array}$$

assign to player I the set:

$$\{0, 1, \ldots, x_0 - 1\} \cup \{y_0, y_0 + 1, \ldots, x_1 - 1\} \cup \cdots \cup \{y_n, y_n + 1, \ldots, x_{n+1} - 1\} \cup \cdots,$$

and assign to player II the complement of this set. Each player wins if their set lies in a fixed non-principal ultrafilter on $\omega$, which we will call $U$. Then suppose player I has a winning strategy.

Player I first plays $x_0$ according to her winning strategy. Then player II is able to play the move that player I would have played in response to player II playing, for example, $x_0 + 1$, call this $x_1'$. Now player I must respond to this move. Player II can now play player I's winning strategy against this move, and so on. So player II has stolen player I's winning strategy, and it only differs from the set that player I would have constructed by playing this strategy below $x_0$. So the difference between the set Player II constructs, and the set Player I *would have* constructed is finite. Therefore player II's set must also lie in $U$. So player I cannot have a winning strategy.

If we suppose player II has a winning strategy, player I can perform a very similar trick. Therefore, neither player can have a winning strategy. □

So we know that finite games *are* determined. We also know that not all sets are determined; however the non-determined payoff set we have constructed requires the axiom of choice, and is somewhat pathological. Aside

[1] This is the only place in the proof of a non-determined set where we use the axiom of choice, so if every game on $\omega$ were determined there would be no non-principal ultrafilter on $\omega$, contradicting the axiom of choice.

from determined and non-determined, there are sets which ZFC cannot prove are determined, but also cannot prove are non-determined. Thus, we can stipulate that they are determined without contradiction and see the consequences thereof. This will ultimately lead us into the world of large cardinals, especially $0^{\#}$ (*zero sharp*) which we will define in section 3. For now, we turn to such a class of sets whose determinacy is independent of ZFC: analytic sets.

# 2 Analytic Determinacy

## 2.1 The Borel Hierarchy

Before we can begin discussing analytic sets, which arise in the *projective hierarchy*, we will first expound on their (hopefully much more familiar) conceptual precursors, the Borel sets and their *Borel hierarchy*.

**Definition 2.1** (Borel hierarchy)**.** Every Borel set on a topology $X$ can be located in a hierarchy which is built inductively, beginning with

$$\boldsymbol{\Sigma}^0_1 := \{A \subseteq X \,:\, A \text{ is open}\}$$

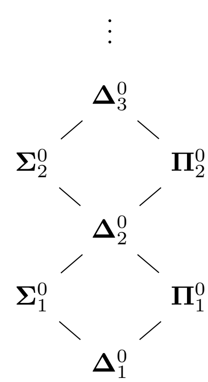
Then for any ordinal $\alpha$, we define

$$\boldsymbol{\Pi}^0_\alpha := \{X \setminus A \,:\, A \in \boldsymbol{\Sigma}^0_\alpha\},$$

so that, for example, $\boldsymbol{\Pi}^0_1$ is the collection of closed subsets of $X$. Then once we've defined $\boldsymbol{\Pi}^0_\beta$ for all $\beta < \alpha$, we can define

$$\boldsymbol{\Sigma}^0_\alpha := \left\{ A \,:\text{there exists } (A_i)_{i\in\omega} \text{ s.t. } A = \bigcup_{i\in\omega} A_i, \text{ and each } A_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta \right\}.$$

Then $\boldsymbol{\Delta}^0_\alpha := \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$, so that the Borel hierarchy looks like:



with inclusions of lower sets in higher sets. It is not difficult, but takes us too far afield, to prove that almost every line in the above diagram can be replaced with an inclusion. As an exception, note that we cannot prove $\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1 \subseteq \mathbf{\Delta}^0_2$; in fact this is not true for a general topological space but is a property of the space itself. We call these properties $\mathrm{F}_\sigma$ in the former case (or $\mathrm{G}_\delta$ in the latter case, but by taking complements these properties are equivalent).

**Proposition 2.2.** *The Borel hierarchy terminates at $\omega_1$, so $\mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Pi}^0_{\omega_1}$.*

*Proof.* We demonstrate that $\mathbf{\Pi}^0_{\omega_1} \subseteq \mathbf{\Sigma}^0_{\omega_1}$, and vice versa. It is enough to show that

$$\mathbf{\Sigma}^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha.$$

In order to show this equality holds, we show containment both directions. One direction of this claim is trivial by the definition of $\mathbf{\Sigma}^0_{\omega_1}$. The other direction is shown by taking $X \in \mathbf{\Sigma}^0_{\omega_1}$ which can be written

$$X = \bigcup_{n\in\omega} X_n,$$

where $X_n$ lies in $\mathbf{\Pi}^0_{\beta_n}$. Then since each $\beta_n$ is countable, and we are taking a countable union, the union of these ordinals is $\beta$ for some countable $\beta$, and thus $X$ lies in $\mathbf{\Sigma}^0_{\beta+1}$ which is contained in $\mathbf{\Pi}^0_{\beta+2}$, and thus contained in the union of $\mathbf{\Pi}^0_\alpha$ for countable $\alpha$. □

**Remark.** Note that the proof we just gave relies on the regularity of $\omega_1$, and the fact that we are only taking *countable* unions, and a countable union of ordinals below $\omega_1$ must remain below $\omega_1$.

The Borel hierarchy can be defined on any topological space, $X$, though in this essay, our focus will be the Baire space.

**Definition 2.3** (Baire space)**. Baire space** consists of the set ${}^\omega\omega$, with the topology generated by a base[2], $\mathcal{B}$, of finite sequences. That is to say:

$$X \in \mathcal{B} \Leftrightarrow \exists s \in {}^{<\omega}\omega,\ X = \{t \in {}^\omega\omega \,:\, (t \restriction |s|) = s\}.$$

Baire space is an $\mathrm{F}_\sigma$ (equivalently $\mathrm{G}_\delta$) topological space, so the Borel hierarchy on ${}^\omega\omega$ has inclusions $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_\beta$ for all ordinals $\alpha < \beta$.

It was proved in 1953 by Gale and Stewart that if a payoff set $A$ is an open or closed subset of ${}^\omega\omega$, then the game $G(A)$ is determined. Over the next twenty years, progress was made in demonstrating that incrementally higher levels of

[2]The choice of base will be important later.

the Borel hierarchy were determined, getting up to $\mathbf{\Sigma}^0_4$ in 1972. Then, in 1975, D. A. Martin proved that for every Borel set $A$, $G(A)$ is determined[3][12]. Over the same period, the study of large cardinal axioms, and development of the large cardinal hierarchy also blossomed. These large cardinal axioms were utilised in order to obtain determinacy conditions for sets more "complicated" than those in the Borel hierarchy. In order for us to follow beyond the realms of Borel determinacy, we first need to discuss what lies beyond the Borel hierarchy.

## 2.2 The Projective Hierarchy

The projective hierarchy starts where the Borel hierarchy stops. We already know that the Borel hierarchy terminates at $\omega_1$, so repeated utilisation of countable unions, complements or countable intersections will take us no further. Thus we need to introduce a new method for constructing classes of sets within topological spaces. We begin with a general definition, so that we have the language to discuss these constructions.

**Definition 2.4** (Pointclass)**.** A **pointclass** $\Gamma$ is an operation which maps a topological space $X$ to a collection of subsets of $X$.

At this level of generality, we can only say that pointclasses are class functions, not necessarily even definable class functions (although every pointclass we will see is a definable class function). So $\mathbf{\Sigma}^0_1$ (respectively $\mathbf{\Pi}^0_1$) constitutes a pointclass which maps a topological space $X$ to the collection of its open (respectively closed) subsets. For every $\alpha \leqslant \omega_1$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ constitute a pointclass.

The discerning reader will have noticed that every pointclass thus far mentioned has been written in boldface, this is not for either arbitrary or aesthetic reasons, but conveys a meaning which we will now establish.

**Definition 2.5** (Boldface Pointclass)**.** We say that a pointclass is **boldface** if it is closed under preimages of continuous functions. That is, whenever $f : X \to Y$ is continuous, and $A \in \Gamma(Y)$, then $f^{-1}(A) \in \Gamma(X)$.

Of course, every pointclass so far discussed is boldface in the technical, as well as notational, sense of the term. This definition, and the observation that every level in the Borel hierarchy is closed under preimages of continuous functions prompts an obvious question. Is every level of the Borel hierarchy closed under *images* of continuous functions? The answer is clearly no, for example $f(x) = e^{-x}$ will map the $\mathbf{\Pi}^0_1$ set[4] $[0, \infty)$ to the set $(0, 1]$.

So levels within the Borel hierarchy are not, in general, closed under images of continuous functions. But we can make an even stronger claim: the hierarchy of all Borel sets is not closed under images of continuous functions. This is due

[3]Which is the best determinacy result possible for the projective hierarchy within ZFC.
[4]In the usual topology on $\mathbb{R}$.

to a result of Suslin, which can be found in [14, Cor. 4.3] Motivated by this, we can define a new type of pointclass.

**Definition 2.6** (Projected pointclass)**.** If $\Gamma$ is a pointclass on $X$, then we define the **projected pointclass** $\exists^{^\omega\omega}\Gamma(X)$ by:

$$A \in \exists^{^\omega\omega}\Gamma(X) \Leftrightarrow \exists B \in \Gamma(X \times {}^\omega\omega) \text{ s.t. } A = p[B],$$

where $p$ is given by:

$$\begin{aligned} p : X \times {}^\omega\omega &\to X \\ (x, y) &\mapsto x \end{aligned}$$

Utilising this definition, we can establish a new hierarchy built not by taking iterated unions and complements, but rather by iterated projections and complements.

**Definition 2.7** (Projective hierarchy)**.** The **projective hierarchy** is defined by setting $\mathbf{\Pi}^1_0 = \mathbf{\Pi}^0_1$. Then, once $\mathbf{\Pi}^1_n$ is defined, we can define $\mathbf{\Sigma}^1_{n+1} = \exists^{^\omega\omega}\mathbf{\Pi}^1_n$. To complete the iterative definition, we set $\mathbf{\Pi}^1_{n+1} = X \setminus \mathbf{\Sigma}^1_{n+1}$.

We call the elements of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ the *analytic* and *co-analytic* sets respectively. Then we can state the earlier theorem of Suslin more precisely: there is an analytic set which is not Borel. Now we can officially state what we mean by analytic determinacy:

**Definition 2.8** (Analytic Determinacy)**.** Analytic determinacy, also written $\mathrm{Det}(\mathbf{\Sigma}^1_1)$, is the claim that every game $G(A)$, where $A$ is an analytic set, is determined.

## 2.3 The Lightface Hierarchy

We have seen how the projective hierarchy is a natural extension of the Borel hierarchy, and so occasionally the two are combined and called simply the *boldface hierarchy*. Now that we've established this hierarchy, we will turn to their counterparts, the *lightface* hierarchy. These will turn out to be relevant to our later proof of Harrington's theorem. However, lightface pointclasses rely heavily on some notions from computability theory and descriptive set theory, so we will first give a brief overview of these topics.

**Definition 2.9** (Recursive/Recursively enumerable set)**.** We say that a set $a \subseteq \omega$ is **recursive** (computable) if there is some Turing machine which computes for any $n \in \omega$, whether $n \in a$ or $n \notin a$, and halts in finite time on all inputs. We say that a set $a$ is **recursively enumerable** (computably enumerable) if there is some Turing machine which halts iff $x \in a$ and does not halt otherwise. Similarly, we say that a set $a \subseteq \omega$ is **recursive in** $b$ (or *Turing-reducible to* $b$) if there is some Turing machine with oracle $b$ which computes $a$. In this case, we write $a \leqslant_{\mathrm{T}} b$. This defines a preorder on subsets of $\omega$, whose equivalence classes we call *Turing degrees.*

We can extend these notions in a natural way to relations $R \subseteq \omega \times \omega$ (requiring a Turing machine which evaluates whether $nRm$ for any $n$, $m$) and reals $r \in {}^{\omega}\omega$ (requiring a Turing machine which computes $r(i)$ for any given $i \in \omega$). With this definition in hand, we begin to establish the lightface hierarchy.

**Definition 2.10** (The arithmetical hierarchy)**.** Recall the definition of our topology on ${}^{\omega}\omega$, given in definition 2.3. The base of this topology, $\mathcal{B}$, consists of the open sets corresponding to finite sequences in ${}^{<\omega}\omega$. Then we call an open subset of ${}^{\omega}\omega$ a *recursively enumerable* open subset if it is the union of a recursively enumerable collection of sets in $\mathcal{B}$. Then

$$\Sigma^1_0 := \{A \subset {}^{\omega}\omega \,:\, A \text{ is a recursively enumerable open subset}\}.$$

Then we say that a set is $\Pi^0_\alpha$ iff it is the complement of a $\Sigma^0_\alpha$ set. Once $\Pi^0_\alpha$ is defined, we can define $\Sigma^0_{\alpha+1}$ as the sets which are a union of a recursively enumerable collection of $\Pi^0_\alpha$ sets. We can also define, just as in the boldface hierarchy $\Delta^0_\alpha := \Sigma^0_\alpha \cap \Pi^0_\alpha$. These form a hierarchy, with inclusions analogous to the Borel hierarchy, where $\Sigma^0_\alpha \subseteq \Sigma^0_{\alpha+1}, \Pi^0_{\alpha+1}$ and $\Pi^0_\alpha \subseteq \Sigma_{\alpha+1}, \Pi^0_{\alpha+1}$.

This segment of the lightface hierarchy is named the *arithmetical hierarchy.* Like the Borel hierarchy, this hierarchy must also terminate at some given level of the ordinal hierarchy by the axiom of replacement. The proof of proposition 2.2 is helpful here. Proposition 2.2 follows from the fact that taking countable unions of countable ordinals keeps us below $\omega_1$. The following proposition follows in a similar manner from the fact that we are taking *computably enumerable* unions of computable ordinals.

**Proposition 2.11.** *The arithmetical hierarchy terminates at $\omega_1^{CK}$, where $\omega_1^{CK}$ is the least non-computable ordinal, that is, the least $\alpha \in \mathrm{Ord}$ for which there is no computable relation on any computable subset of the natural numbers which has order-type $\alpha$.*

**Definition 2.12** (Hyperarithmetical hierarchy)**.** Just as with the Borel hierarchy, there is another level of the lightface hierarchy that comes after the arithmetical hierarchy, which is named the **hyperarithmetical hierarchy**. This is given by defining a subset of ${}^{\omega}\omega$ to be $\Sigma^1_1$ iff it is the projection of a $\Pi^0_1$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$. Then, as in definition 2.7 we say a subset of ${}^{\omega}\omega$ is $\Pi^1_\alpha$ iff it is the complement of a $\Sigma^1_\alpha$ set. Finally, a subset of ${}^{\omega}\omega$ is $\Sigma^1_{\alpha+1}$ iff it is the projection of a $\Pi^1_\alpha$ subset of ${}^{\omega}\omega \times {}^{\omega}\omega$.

We call the collection of $\Sigma^1_1$ sets the *lightface analytic sets*, and the collection of $\Pi^1_1$ sets the *lightface co-analytic sets.* [7, Def. 25.1]

**Remark.** Note that all of these pointclasses are reliant on the particular notion of recursivity being employed. Therefore, we could give a Turing machine an oracle and follow an identical series of definitions and results to obtain a *relativised* version of the lightface hierarchy. For example, we say that a set $A$ is $\Sigma^0_1(a)$ where $a \subseteq \omega$ if $A$ can be written as a union of a collection of basic open sets which are recursive in $a$. This will change certain results, for example the relativised arithmetical hierarchy will not terminate at the first uncomputable ordinal, but rather the first ordinal which is uncomputable relative to $a$. It is not difficult to see that we could encode any countable collection of basic open sets with subsets of $\omega$, so $\mathbf{\Sigma}^0_1 = \bigcup_{a \in {}^\omega\omega} \Sigma^0_1(a)$, and the analogous result is true for all levels of the arithmetical hierarchy, and then immediately from the definition also into the hyperarithmetical hierarchy.

# 3 Sharps

## 3.1 Indiscernibles

Sharps originated as model-theoretic constructions [8, beginning of §9]. Ehrenfeucht and Mostowski first isolated the notion of indiscernibles in order to facilitate the construction of models with large numbers of automorphisms [3]. Silver, in his 1966 PhD thesis, cast ordinals in the role of indiscernibles for $L$, and thereby manged to obtain marvellous results about $L$ and about the difference between $L$ and $V$[17]. The presentation of sharps we give is influenced by [8, §9] and [7, §18]

**Definition 3.1** (Indiscernibles)**.** We say that a set $X$ is a set of (order-)**indiscernibles** for a structure $\mathcal{M}$ iff for any sentence $\varphi$

$$\mathcal{M} \vDash \varphi(x_{i_1}, \dots, x_{i_n}) \Leftrightarrow \mathcal{M} \vDash \varphi(x_{j_1}, \dots, x_{j_n}),$$

where $x_{i_1} < \dots < x_{i_n}$ and $x_{j_1} < \dots < x_{j_n}$ all lie in $X$.

**Definition 3.2** (Skolem function)**.** For a theory $T$ in a language $\mathcal{L}$, and a given sentence $\varphi$ in $\mathcal{L}$, we say that a function $t_\varphi$ is a **Skolem function** for $\varphi$ if:

$$T \vdash \forall \bar{y}(\exists x \varphi(x, \bar{y}) \Rightarrow \varphi(t_\varphi(\bar{y}), \bar{y})).$$

We say that a theory has Skolem functions if there is a Skolem function for every sentence in the language.

When there are multiple possible assignments of a Skolem function, we will define that the Skolem function evaluates to the minimal element satisfying the statement. This is well-defined since we will be working with $L$, and there is a definable well-ordering, $<_L$, on $L$.

**Theorem 3.3** (Ehrenfeucht-Mostowski)**.** *Let $T$ be a theory (with Skolem functions) such that there exists $\mathcal{M} \vDash T$, with $|\mathcal{M}|$ infinite. Then let $(X, <)$ be any* *infinite set with a linear order. There exists a model of $T$ which contains $X$, so that $X$ is a set of indiscernibles.*

*Proof sketch.* This is shown by expanding the language of $T$ with constant symbols, and adding statements to $T$ which entail that any sentence which holds for some $n$ increasing elements in $X$ holds for all $n$ increasing elements in $X$. Then using Ramsey's theorem, we can show that finite subtheories of $T$ have models, and finally applying compactness proves the theorem. □

**Definition 3.4** (Skolem hull)**.** The Skolem hull of a set $A \subseteq \mathcal{M}$ is

$$\{t_\varphi(a_1, \ldots, a_n) \,|\, a_1, \ldots, a_m \in A\},$$

where $t_\varphi$ are Skolem functions for formulae in $\mathcal{M}$.

**Definition 3.5** (Ehrenfeucht-Mostowski Blueprint)**.** An **Ehrenfeucht-Mostowski blueprint** (or EM blueprint) is the theory of $(L_\delta, \in, c_i)_{i<\omega}$ in the language $\mathcal{L} = \{\in\} \cup \{c_i : i < \omega\}$, where $c_i$ are constant functions representing ordinal indiscernibles for $L_\delta$.

An EM blueprint is *the set of sentences* of such a theory, however we will often use EM blueprint to mean the set of Gödel codes of such sentences.

If $T$ is an EM blueprint, we will use $T^-$ to refer to the same theory in a language without constants. We obtain this theory by removing any sentence in our EM blueprint which contains an occurence of $c_i$ for all $i$.

**Lemma 3.6.** *For any $\alpha \in \mathrm{Ord}$ there is a unique (up to isomorphism) structure, which we denote by $\mathcal{M}(T, \alpha)$ which models $T^-$ and has*

*(i) A set $X$ such that*

$$\forall x \in X,\ \mathcal{M}(T, \alpha) \vDash x \in \mathrm{Ord},$$

*and such that $X$ is a set of indiscernibles for $\mathcal{M}$ of order-type $\alpha$. Additionally, $\varphi(x_{i_1}, \ldots, x_{i_n})$ holds in $\mathcal{M}$ iff $\varphi(c_0, c_1, \ldots, c_n) \in T$.*

*(ii) The Skolem hull of $X$ is $\mathcal{M}$. That is, we can get any element in $\mathcal{M}$ from applying a Skolem function to some collection of elements in $X$.*

*Proof.* Theorem 3.3 gives that there is a model satisfying condition (i). Taking the Skolem hull of $X$ within this model gives a model satisfying condition (ii). To show uniqueness, supppose there were two such models, $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$, which satisfy these conditions. We can define an order-isomorphism between $X$ and $Y$, and since every term in both models can be generated from elements in $X$ and $Y$ respectively using Skolem functions by condition (ii), we can extend this to an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. So such an $\mathcal{M}$ must be unique up to isomorphism. □

### 3.2 The conditions for "$0^\#$ exists"

When we ultimately state "$0^\#$ exists" formally, it will *technically* be a claim about the existence of a particular set. This set will consist of the Gödel encodings of true sentences in a type of EM blueprint, so in fact the existence of this set will be shorthand to denote the existence of such a theory. However, this will not be just any EM blueprint, therefore in this section we will outline the conditions we need to impose on the theory in order for "$0^\#$ exists" to hold.

If we suppose that $\mathcal{M}(T,\alpha)$ is well-founded, then the transitive collapse of $\mathcal{M}(T,\alpha)$ is $L_\delta$ for some $\delta > \omega$ by Gödel's condensation lemma. Further, we need only stipulate that $\mathcal{M}(T,\alpha)$ is well-founded for all countable ordinals.

**Lemma 3.7** (The well-founded condition)**.** *Suppose that*

$$\mathcal{M}(T,\alpha) \textit{ is well-founded for countable ordinals,} \tag{$*$}$$

*then for all* $\alpha \in \mathrm{Ord}$*,* $\mathcal{M}(T,\alpha)$ *is well-founded. We call* $(*)$ *the "well-founded condition."*

*Proof.* Suppose $\alpha$ is an uncountable ordinal which is not well-founded, take the collection of elements $v_i$ witnessing this, so that $\forall i \in \omega$, $v_{i+1} E v_i$. Then by lemma 3.6(ii) every element $\mathcal{M}(T,\alpha)$ can be written as $t_\varphi(x_1,\ldots,x_n)$ for some $x_i \in X$. Therefore, for each $v_i$ we have

$$v_i = t_\varphi(x_{j_1},\ldots,x_{j_n}).$$

Let $X'$ be the collection of every $x_j$ which occurs in any term defining a $v_i$, then $X'$ is countable, since each term contains only a finite number of elements, and there are $\aleph_0$ many terms in the descending sequence. Then if we let $\mathcal{N}$ be the Skolem hull of $X'$ in $\mathcal{M}(T,\alpha)$, we have that $\mathcal{N}$ is ill-founded, but isomorphic to $\mathcal{M}(T,\beta)$ where $\beta$ is the order-type of $X'$. This gives us a countable $\beta$ such that $\mathcal{M}(T,\beta)$ is ill-founded. Contradiction. □

**Lemma 3.8** (The unbounded condition)**.** *For* $\mathcal{M}(T,\alpha)$*, the set of indiscernibles,* $X$*, is unbounded (cofinal) in the ordinals of* $\mathcal{M}(T,\alpha)$ *iff for every Skolem term* $t_\varphi$*, the following sentence is in* $T$*:*

$$t_\varphi(c_0,\ldots,c_n) \in \mathrm{Ord} \to t_\varphi(c_0,\ldots,c_n) < c_{n+1} \tag{$\bullet$}$$

*If every such sentence is in* $T$*, we say* $T$ *satisfies the unbounded condition.*

*Proof.* First we prove that $(\bullet)$ being in $T$ implies that $X$ is unbounded. Let $\mathcal{M}(T,\alpha) \models a \in \mathrm{Ord}$. Then there is some Skolem function $t_\varphi$ and some collection of increasing $x_i \in X$ such that $a = t_\varphi(x_0,x_1,\ldots,x_n)$ by lemma 3.6(ii). Then we have $a < x_{n+1}$ by hypothesis, so there is some element of $X$ greater than $a$ for any ordinal $a$ of $\mathcal{M}(T,\alpha)$.

For the other direction, if $X$ is unbounded, then let $a = t_\varphi(x_0,\ldots,x_n)$

be any Skolem function applied to a collection of increasing terms $x_i \in X$. If $\mathcal{M}(T,\alpha) \vDash a \notin \mathrm{Ord}$, then we're trivially done. Otherwise, $a \in \mathrm{Ord}$, and then by unboundedness we have

$$\exists x_k \in X \,:\, a, x_n < x_k.$$

Then by indiscernibility we have $a, x_n < x_{n+1}$. So (•) holds, and therefore must lie in $T$. □

**Definition 3.9** (Remarkable)**.** For $\alpha > \omega$ a limit ordinal, we say that $\mathcal{M}(T,\alpha)$ with a set $X$ of indiscernibles is **remarkable** if the Skolem hull of $\{x_i \,:\, i \in \omega\}$ contains all ordinals in $\mathcal{M}(T,\alpha)$ below $x_\omega$.

**Lemma 3.10** (The remarkable condition)**.** *The model $\mathcal{M}(T,\alpha)$ is remarkable for all $\alpha > \omega$ iff for every Skolem term $t_\varphi$, $T$ contains the sentence*

$$t(c_0,\ldots,c_{m+n}) < c_n \to t(c_0,\ldots,c_{m+n}) = t(c_0,\ldots,c_{n-1},c_{i_1},c_{i_2},\ldots,c_{i_{m+1}}). \tag{+}$$

*where $n \leqslant i_1 < i_2 < \cdots < i_{m+1}$. If every such sentence is in $T$, we say $T$ satisfies the remarkable condition.*

*Proof.* If $\mathcal{M}(T,\alpha)$ is remarkable for $\alpha > \omega$ a limit ordinal, then let $x_1 < \cdots \cdots < x_n$ be the first $n$ members of $X$, and let $x_{i_1} < \cdots < x_{i_{m+1}}$ be such that $i_1 = \omega$. Set $a := t_\varphi(x_1,\ldots,x_n,x_{i_1},\ldots,x_{i_{m+1}})$ for some Skolem function $t_\varphi$. If $a$ is not an ordinal or is an ordinal greater than or equal to $x_{i_1}$ we are trivially done. Otherwise, $a$ is an ordinal below $x_\omega$ so remarkability implies that $a$ lies in the Skolem hull of $\{x_i \,:\, i \in \omega\}$. By the definition of the Skolem hull, there exists a different Skolem function $t_\psi$ such that $a = t_\psi(y_1,\ldots,y_k)$ for $y_i \in \{x_i \,:\, i \in \omega\}$. Then by indiscernibility, and the fact that

$$\mathcal{M}(T,\alpha) \vDash t_\varphi(x_1,\ldots,x_n,x_{i_1},\ldots,x_{i_{m+1}}) = t_\psi(y_1,\ldots,y_k),$$

we can interchange $x_{i_1},\ldots,x_{i_{m+1}}$ with any other indiscernibles which have the same order-relation to $x_1,\ldots,x_n$, and (+) remains true.

For the other direction, we will show that if (+) holds, then for any limit ordinal $\alpha$, every ordinal below $x_\alpha$ lies in the Skolem hull of $\{x_\beta \,:\, \beta < \alpha\}$. Let $a$ be such that $\mathcal{M}(T,\alpha) \vDash a \in \mathrm{Ord}$, and $a < x_\alpha$. Then by lemma 3.6(ii), we have that $a = t(x_1,\ldots,x_n,x_{i_1},\ldots,x_{i_{m+1}})$ for $x_1 < \cdots < x_n < x_\alpha$, and $x_\alpha = x_{i_1} < \cdots < x_{i_{m+1}}$. Then since (+) holds and using indiscernibility, $a < x_{i_1}$ means we can replace $x_{i_1},\ldots,x_{i_{m+1}}$ with $x_{n+1},\ldots,x_{m+n+1}$ and since $\alpha$ is a limit ordinal, we've only used terms $x_\beta$ for $\beta < \alpha$, so $a$ lies in the Skolem hull of $\{x_\beta \,:\, \beta < \alpha\}$ in $\mathcal{M}(T,\alpha)$. □

**Remark.** Often in the literature, remarkability is defined so as to include the unboundedness condition, so that "$0^\#$ exists" is equivalent to the existence of a "well-founded, remarkable EM blueprint." Here we have chosen to separate remarkability and unboundedness so that it is clearer that there are three conditions which an EM blueprint must satisfy for "$0^\#$ exists" to hold, and to analyze the unboundedness condition in isolation.

## 3.3 The sharp of a set

Now we suppose that there exists an EM blueprint satisfying the well-founded, unbounded, and remarkability conditions.

**Proposition 3.11.** *The structure $\mathcal{M}(T, \kappa)$ is isomorphic to $L_\kappa$ when $\kappa$ is an uncountable cardinal.*

*Proof.* Since $\mathcal{M}(T, \kappa)$ is a model of $T^-$, we have that it is elementarily equivalent to $L_\delta$ for some $\delta$. Then by Gödel's condensation lemma, there exists some $\alpha$ such that $\mathcal{M}(T, \kappa) \cong L_\alpha$. Since $X \subseteq L_\alpha$ is a set of ordinals of order-type $\kappa$, we must have $\alpha \geqslant \kappa$. If $\alpha > \kappa$, then since $X$ is unbounded in $\alpha$, there must be some ordinal larger than $\kappa$ in $X$ which is the $\gamma^{\text{th}}$ ordinal in the ordering of $X$ for $\gamma < \kappa$ (since $X$ has order-type $\kappa$). Then from the proof of lemma 3.10, we know that the Skolem hull of $\{x_i : i < \gamma\}$ contains all ordinals below $x_\gamma$. Therefore $\kappa$ is a subset of this Skolem hull, so the size of this set must be greater than or equal to $\kappa$. On the other hand, the Skolem hull is generated by countably many Skolem terms each containing only $\gamma < \kappa$ possible ordinals, so the Skolem hull must have cardinality $< \kappa$, but include $\kappa$ as a subset, which is absurd. Therefore $\mathcal{M}(T, \kappa) \cong L_\kappa$. □

Then also, using the following two lemmas, we will be able to see that any uncountable cardinal $\kappa < \alpha$ is an indiscernible in $\mathcal{M}(T, \alpha)$.

**Lemma 3.12.** *If $\kappa < \lambda$ are both uncountable cardinals. Let $X$ be the set of indiscernibles for $\mathcal{M}(T, \lambda) \cong L_\lambda$ and $Y$ the set of indiscernibles for $\mathcal{M}(T, \kappa) \cong L_\kappa$, then $X \cap L_\kappa = Y$.*

*Proof.* Omitted. See [7, Lem. 18.14] □

**Lemma 3.13.** *The set of indiscernibles of $L_\alpha$ are closed and unbounded in $\alpha$.*

*Proof.* We have that the set of indiscernibles of $L_\alpha$ is unbounded in $\alpha$ from 3.8. A proof of closure can be found in [7, Lem. 18.12] □

Using this result, we can prove that any uncountable cardinal must be an indiscernible.

**Lemma 3.14.** *If $\omega_1 < \kappa < \lambda$ are cardinals, then $\kappa$ is an indiscernible of $L_\lambda$.*

*Proof.* For any uncountable cardinals $\kappa < \lambda$, we have that the set of indiscernibles for $L_\lambda$ which lie in $L_\kappa$ are a closed unbounded subset of $\kappa$. Therefore $\kappa$ must lie in the set of indiscernibles for $L_\lambda$ by closure. □

Now that we have these conditions, definitions and lemmas, we can finally state the principal definition of this section.

**Definition 3.15.** We say "$0^{\#}$ exists" iff there exists a well-founded, unbounded, remarkable EM blueprint. If such a set exists then we can define it as

$$\{\ulcorner\varphi\urcorner : L_{\aleph_\omega} \vDash \varphi(\aleph_1, \ldots, \aleph_n)\},$$

since by proposition 3.11, $L_{\aleph_\omega}$ is isomorphic to $\mathcal{M}(T, \aleph_\omega)$, and by lemma 3.14 each $\aleph_i$ for $i \geqslant 1$ is an indiscernible of $L_{\aleph_\omega}$. Therefore by indiscernibility and closure under taking the Skolem hull, any statement modelled by $L_{\aleph_\omega}$ will be equivalent to one of the above form. Moreover, $\varphi(c_1, \ldots, c_n)$ is in $T$ if and only if $L_{\aleph_\omega} \vDash \varphi(\aleph_1, \ldots, \aleph_n)$, so this EM blueprint is unique.

**Remark.** In fact, all the preceding definitions, lemmas and conditions apply equally well when $a \subseteq \omega$, and we take $L(a)$ to be the union of the *relativised constructible hierarchy*, where instead of beginning with the empty set, we begin with the transitive closure of $a$. An EM blueprint with this definition is called $a^{\#}$.

**Remark.** This definition of "$0^{\#}$ exists" or "$a^{\#}$ exists" as a claim about the existence of a particular set is a common but somewhat unfortunate one. This is because it is always possible (in ZFC) to prove that the set of sentences $\{\ulcorner\varphi\urcorner : L_{\aleph_\omega} \vDash \varphi(\aleph_1, \ldots, \aleph_n)\}$ exists. The crux of definition 3.15 is that the set which is defined constitutes a *well-founded, unbounded, remarkable EM blueprint* – this is *not* provable in ZFC alone. When this is the case we can also define the particular set in definition 3.15 and say that this set "is" $0^{\#}$.

## 3.4 An equivalent definition of "$0^{\#}$ exists"

We will often taken an alternative definition of the existence of $0^{\#}$, though the origins of the concept lie in model theory as accounted above. We will state but not prove that the two definitions are equivalent.

**Theorem 3.16.** $0^{\#}$ *exists iff there exists a nontrivial elementary embedding* $j : L \to L$.

*Proof.* Omitted. See [7, Thm 18.20] or [8, Thm 9.17]. □

And in fact more is true, the existence of $a^{\#}$ for any real $a \subseteq \omega$ is equivalent to the existence of an elementary embedding $j : L(a) \to L(a)$. We can even define "$A^{\#}$ exists" for $A$ being any set of ordinals iff there is a closed unbounded set $X$ of indiscernibles for $L(A)$ such that the Skolem hull of $X \cup A$ generates $L(A)$[5]

Using this equivalent elementary embedding definition of $0^{\#}$, we are able to state the following corollary.

[5]This works as long as $A$ is a transitive set. If $A$ is not transitive, then $A^{\#}$ exists iff $\mathrm{tcl}(A)^{\#}$, the sharp of the transitive closure of $A$, exists.

**Corollary 3.17.** *Suppose there is a nontrivial elementary embedding $j : L_\alpha \to L_\beta$, with $\gamma = \mathrm{crit}(j) < |\alpha|$, then $0^\#$ exists.*

*Proof sketch.* Form an ultrapower of $L$ using the $L$-ultrafilter[6] $U = \{X \subset \gamma : \gamma \in j(X)\}$. Then, using a lemma found in [7, Lem. 18.22], we have the necessary conditions on the ultrafilter so that $\gamma$ is a critical point of the embedding $j_U : L \to L/U$ (i.e. the ultrafilter is $\gamma$-complete). Then it suffices to prove that the ultrapower is well-founded, so we can take the transitive collapse. Since $j$ is an elementary embedding, and the transitive collapse of $L/U$ is $L$, we have that the mapping from $L$ to $L$ provided by the composition of $j_U$ and the transitive collapse of $L/U$ is an elementary embedding from $L$ into itself.

So all we need to show in order to demonstrate that this implies the existence of $0^\#$ is that the ultrapower is well-founded. However, the proof proceeds almost exactly as the proof of proposition 4.11, so we will postpone the rest of the proof until then. □

**Remark.** As a final note on sharps, an important consequence of $0^\#$ is that it is incompatible with $V = L$. From the results we already have, it is not difficult to prove this; if $V = L$ and "$0^\#$ exists" hold, then the existence of an elementary embedding from $L$ to itself would also mean that there is an elementary embedding from $V$ to itself, contradicting Kunen's inconsistency theorem (see e.g. [7, Thm 17.7] or [8, thm 23.12]).

Now that we have established the definitions of both $0^\#$ and analytic determinacy, as well as the background material for these results, we turn our attention to establishing the implication of the existence of $0^\#$ from analytic determinacy.

# 4 Harrington's Theorem

**Theorem 4.1** (Harrington's Theorem [6])**.** *If $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$ holds, then $0^\#$ exists.*

In order to prove that analytic determinacy implies the existence of $0^\#$, we first need to prove that analytic determinacy implies a statement which we will call Harrington's principle (or $\mathsf{HP}$, defined in 4.8); then we can prove that $\mathsf{HP}$ implies the existence of $0^\#$. In fact, every known proof of Harrington's Theorem uses this proof tactic [2, p.viii]. For example Sami's proof [15] differs significantly from the original proof by Harrington, yet still utilises $\mathsf{HP}$ as a critical intermediate step.

Yong Cheng [2] analysed both the proof of "$0^\#$ exists" and the consequences of $\mathsf{HP}$ in weaker theories. Namely, the theories corresponding to second-, third-, and fourth-order arithmetic:

[6] A collection of sets such that $L \models U$ is an ultrafilter.

**Definition 4.2** (Higher-order arithmetic)**.** The following are the set-theoretic systems corresponding (via codings) to systems of higher-order arithmetic [2, Def 1.9]:

(i) $\mathsf{Z}_2 := \mathsf{ZFC}^- +$ "every set is countable".

(ii) $\mathsf{Z}_3 := \mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists" + "every set has cardinality $\leqslant \beth_1$"

(iii) $\mathsf{Z}_4 := \mathsf{ZFC}^- +$ "$\mathcal{P}(\mathcal{P}(\omega))$ exists" + "every set has cardinality $\leqslant \beth_2$"

Cheng proved the following result.

**Theorem 4.3.** $\mathsf{Z}_3 + \mathsf{HP} \nvdash$ *"$0^\#$ exists".*

*Proof.* Given in [2, §2.2, §2.3]. □

However, it is not the case that we need the power-set axiom to prove that $\mathsf{HP} \Rightarrow$ "$0^\#$ exists", and in fact Cheng managed to show:

**Theorem 4.4.** $\mathsf{Z}_4 + \mathsf{HP} \vdash$ *"$0^\#$ exists".*

*Proof.* Given in [2, §2.4]. □

Woodin conjectured [2, p. 27] that $\mathrm{Det}(\mathbf{\Sigma}^1_1) \Rightarrow$ "$0^\#$ exists" is provable in $\mathsf{Z}_2$, and it is known that the reverse implication of Harrington's theorem *is* provable in $\mathsf{Z}_2$ [2, §3.1]. However, this work by Cheng demonstrates that if Woodin's conjecture were true, the proof in $\mathsf{Z}_2$ would not be able to proceed via $\mathsf{HP}$ as with all presently known proofs of Harrington's theorem, since even the stronger theory $\mathsf{Z}_3$ cannot prove that $\mathsf{HP}$ implies "$0^\#$ exists."

In any case, we too will go via $\mathsf{HP}$, and we begin by proving the second part of this claim, that $\mathsf{HP}$ implies $0^\#$ exists

## 4.1 Harrington's Principle implies $0^\#$ exists

Before we can formally state Harrington's principle, we will need to provide an explanation of admissible sets. The concept of an admissible sets arises naturally in the study of a weak form of set theory called Kripke-Platek set theory, hereafter $\mathsf{KP}$ [1, Ch. 1, Defs 2.2, 2.5].

**Definition 4.5** ($\mathsf{KP}$)**.** $\mathsf{KP}$ is the theory in the language $\mathcal{L} = \{\in\}$ (potentially augmented by a unary predicate $a$), which arises from the universal closure of the following axioms:

(i) *Extensionality:* $\forall x(x \in a \leftrightarrow x \in b) \to a = b$.

(ii) *Foundation:* $\exists x\, \varphi(x) \to \exists x\, [\varphi(x) \wedge \forall y \in x\, \neg\varphi(y)]$ for each formula $\varphi(x)$ with no free occurences of $y$.

(iii) *Pairing:* $\exists a\,(x \in a \wedge y \in a)$

(iv) *Union:* $\exists b\, \forall y \in a\, \forall x \in y\,(x \in b)$

(v) $\Delta_0$ *Separation:* $\exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$ for all formulas $\varphi$ which are $\Delta_0$ and without free occurrences of $b$.

(vi) $\Delta_0$ *Collection:* $\forall x \in a\, \exists y\, \varphi(x,y) \to \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x,y)$ for all formulas $\varphi$ which are $\Delta_0$ and without free occurrences of $b$.[7]

With this definition, we can define admissible ordinals as follows.

**Definition 4.6** (Admissible Ordinal/Set)**.** We say that a set $M$ is an **admissible set** if $M \vDash \mathsf{KP}$. We say that an ordinal $\alpha$ is an **admissible ordinal** if $(L_\alpha, \in)$ models $\mathsf{KP}$. We say that an ordinal is $a$-admissible iff $(L_\alpha[a], a \cap L_\alpha[a], \in)$ is a model of $\mathsf{KP}$ with a unary predicate.

**Definition 4.7** (Least $a$-admissible ordinal)**.** For $a \subset \omega$, write $\omega_1(a)$ to denote the **least $a$-admissible ordinal** (greater than $\omega$) [1, Cor. 2.2].

Now that we have defined admissible sets, we can state Harrington's principle. There are two equivalent formulations, of which we will use only one:

**Definition 4.8** (Harrington's Principle, [6])**.** We say that **Harrington's principle** (or $\mathsf{HP}$) holds if there exists an $a \subseteq \omega$, such that if $\alpha$ is $a$-admissible, then $\alpha$ is a cardinal in $L$[8].

**Theorem 4.9.** $\mathsf{HP}$ *implies that there is a nontrivial elementary embedding from $L_{\omega_2}$ to itself with critical point below $\omega_2$.*

Once we have proven this theorem, we will be able to use corollary 3.17 to show that $0^\#$ exists. We follow the exposition at the start of [2, §2.2].

**Lemma 4.10.** $\mathsf{HP}$ *implies that there is an $L$-ultrafilter $U$ on an $L$-cardinal $\kappa$, such that there is an elementary embedding $L_{\omega_2} \to L_{\omega_2}/U$.*

*Proof.* First, let $a \in {}^\omega\omega$ be the real number which witnesses $\mathsf{HP}$. Working in $L[a]$, choose $\eta > \omega_2$ so that $\eta$ is $a$-admissible, and choose an $N \prec L_\eta[a]$ such that: $\omega_2 \in N$; $|N| = \omega_1$; and ${}^\omega N \subseteq N$, meaning that $N$ is closed under countable sequences.

Then since $\omega_2 \in N$ and $|N| = \omega_1$, $N$ can't be a transitive set, so we take the Mostowski collapse of $N$ given by $\pi : N \cong L_\theta[a]$, since we know $N \cong L_\theta[a]$ by a generalization of Gödel's condensation lemma. Let the function

[7]When we augment $\mathsf{KP}$ with a unary predicate, this predicate may appear in the $\Delta_0$ sentences in the separation and collection axiom schema.

[8]The alternative and equivalent statement requires that $\alpha$ is a countable ordinal, however this was not the principle used by Harrington in his original paper.

$j : L_\theta[a] \to N \prec L_\eta[a]$ be the inverse of the collapsing map, with $\mathrm{crit}(j) = \kappa$, guaranteed to exist since $|L_\theta[a]| = |N| = \aleph_1$, so $\omega_2 \notin \mathrm{Ord} \cap L_\theta[a]$, so we must have some $\kappa \in L_\theta[a]$ such that $j(\kappa) > \kappa$. Then since $N$ is an $a$-admissible set, $\theta$ is an $a$-admissible ordinal, and thus $\mathsf{HP}$ gives us $(\theta \text{ is a cardinal})^L$. Now define an ultrafilter on $\kappa$ by

$$U := \{X \subseteq \kappa \mid X \in L \wedge \kappa \in j(X)\}. \qquad (*)$$

Since $\kappa < \theta$ and $\theta$ is a cardinal in $L$, we have $(\kappa^+)^L \leqslant \theta$, and therefore $U \subseteq L_\theta$ and $L \vDash U$ is an ultrafilter on $\kappa$. Using this ultrafilter, take the ultrapower of $(L_{\omega_2}, \in)$ given by

$$f \sim g \Leftrightarrow \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U.$$

Let $[f]$ denote the equivalence class of $f$ in $L_{\omega_2}$, and define

$$L_{\omega_2}/U = \{[f] \mid f : \kappa \to L_{\omega_2}\}$$

We then have an elementary embedding $e : L_{\omega_2} \to L_{\omega_2}/U$ given by $x \mapsto [c_x]$, where $c_x$ is the constant map from $\kappa$ to $x$. □

**Proposition 4.11.** *$(L_{\omega_2}/U, E)$ is well-founded.*

*Proof.* Suppose not, then we have some sequence of functions $(f_n)_{n\in\omega}$ which is infinitely descending, so $\forall n \in \omega$, $f_{n+1} E f_n$, and therefore $\{\alpha : f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$. Define

$$A_n = \{\alpha : f_{n+1}(\alpha) \in f_n(\alpha)\} \in U.$$

Then $A_n \subseteq \mathcal{P}(\kappa)$, and by the definition of $U$, every set in $U$ lies in $L$. Moreover, since $(\kappa^+)^L \leqslant \theta$, $\mathcal{P}(\kappa) \cap L \subseteq L_\theta$, so each $A_n \in L_\theta$, and therefore also in $L_\theta[a]$.

Now, since ${}^\omega N \subseteq N$, the isomorphism between $N$ and $L_\theta[a]$ gives that ${}^\omega L_\theta[a] \subseteq L_\theta[a]$. This means the sequence $(A_n)_{n\in\omega}$ lies in $L_\theta[a]$. By the definition of $U$, we have that $\kappa \in j(A_n)$ for each $n \in \omega$. Therefore $\kappa \in \bigcap_{n\in\omega} j(A_n)$, and since $j$ is elementary, $\kappa \in j(\bigcap_{n\in\omega} A_n)$. So $\bigcap_{n\in\omega} A_n \neq \emptyset$. Then take $\alpha \in \bigcap_{n\in\omega} A_n$, and have

$$\forall n \in \omega \;\; f_{n+1}(\alpha) \in f_n(\alpha).$$

So $L_{\omega_2}$ is ill-founded, which is absurd. Therefore $L_{\omega_2}/U$ must be well-founded. □

So we have that $L_{\omega_2}/U$ is well-founded, $L_{\omega_2} \preceq L_{\omega_2}/U$, and $L_{\omega_2} \vDash V = L$. These together imply that $L_{\omega_2}/U$ also models $V = L$. Therefore, we have $L_{\omega_2}/U \cong L_\gamma$ for some $\gamma$. Then $|L_{\omega_2}/U| \leqslant \aleph_2$, so we must have $\gamma \leqslant \omega_2$. Also, since $L_{\omega_2}$ elementarily embeds into $L_{\omega_2}/U$, we have $\gamma \geqslant \omega_2$. Therefore the elementary embedding from $L_{\omega_2}$ to $L_{\omega_2}/U$ is an elementary embedding from $L_{\omega_2}$ to itself, with critical point $\kappa < |\omega_2|$. Then by corollary 3.17, $0^\#$ exists.

## 4.2 Analytic Determinacy implies HP.

### 4.2.1 Tree forcings

Harrington attributes the following arguments to Steel [18]. We begin with a basic definition:

**Definition 4.12** (Tree)**.** A **tree** is a subset of ${}^{<\omega}\omega$, the set of functions with domain some $n < \omega$ and codomain $\omega$, such that: $\langle\ \ \rangle \in T$, the empty sequence is always in a tree; and so that $\tau \in T$ and $\eta \subseteq \tau$ implies $\eta \in T$, trees are closed under taking prefixes.

From a given tree $T$, we can define a height function:

**Definition 4.13** (Height function of a tree $T$)**.** Let the **height function** of a tree, $H_T : T \to \mathrm{Ord} \cup \{\infty\}$ be defined as follows:

$$H_T(\tau) = \begin{cases} \infty, & \text{if } \tau \text{ in some infinite branch of } T, \\ \sup\{H_T(\eta) + 1 : \eta \in T,\ \tau \subsetneq \eta\}, & \text{otherwise.} \end{cases}$$

**Remark.** We get large $H_T$ values in the ordinals by infinite branch*ings*, not infinite branch*es*; if an element of a tree lies in an infinite branch, then we assign it the "undefined" value $\infty$. Note also that $H_T$ decreases as we go along a branch of the tree. That is, if $\eta$ is a proper extension of $\tau$, then $H_T(\eta) < H_T(\tau)$ (where for convenience we say every ordinal is less than $\infty$, and $\infty < \infty$).

We can identify trees with subsets of $\omega$, and therefore also with elements of ${}^{\omega}\omega$. Then if $\beta \in \mathrm{Im}(H_T)$, $\omega_1(T) > \beta$, since we can construct the successor of $\beta$ using the tree in a computable manner by recursing along $T$[9]. We can define $\omega_1(T)$ naturally through a computable identification of trees with subsets of $\omega$.

**Definition 4.14** (The forcing poset $Q_\alpha$)**.** For $\alpha \in \mathrm{Ord}$, define the forcing poset $(Q_\alpha, \leqslant_\alpha)$ to be the set consisting of elements of the form $(t, h)$, where $t$ is a tree on ${}^{<\omega}\omega$ as defined above, and $h : t \to \omega \cdot \alpha \cup \{\infty\}$ satisfies the following two conditions:

(i) $h(\langle\ \rangle) = \infty$. The empty sequence is mapped to infinity by $h$.

(ii) If $\eta \subsetneq \tau$ then $h(\tau) < h(\eta)$. As the input to $h$ moves "further" along the branches of the tree, $h$ decreases.

As before, we assume that $\infty < \infty$ and for all $\alpha \in \mathrm{Ord}$, $\alpha < \infty$. We define the ordering $\leqslant_\alpha$ naturally by $(t, h) \leqslant_\alpha (t', h')$ iff $t' \subseteq t$ and $h' \subseteq h$.

[9] We define $\omega_1(T)$ for a tree in the same way as we define it for subsets of $\omega$ in definition 4.7 by a computable bijection between trees on ${}^{<\omega}\omega$ and subsets of $\omega$.

All of this setup about trees and height functions was really preamble so that we could introduce a forcing notion which builds on these definitions. This forcing notion is due to Steel, and is characterised as follows. See [18] for more details.

**Definition 4.15** (Steel forcing)**.** The language of Steel forcing consists of:

(i) A constant symbol $\eta$ for each $\eta \in {}^{<\omega}\omega$.

(ii) A unary relation, $T'$, which ought to be interpreted as standing for a tree.

(iii) The logical symbols $\in$, $\neg$ and $\bigwedge_{n\in\omega}$.

The formulae of Steel forcing are built inductively, assigning a rank to each formula as follows:

(i) Atomic formulae are of the form $\eta \in T'$: This is interpreted as a claim about the sequence $\eta$ being in a tree. If $\varphi$ is an atomic formula, then $\text{rank}(\varphi) = 1$.

(ii) If $\varphi$ is a formula, so is $\neg\varphi$: This is interpreted as a claim about a sequence or set of sequences *not* being in a tree. We set $\text{rank}(\neg\varphi) = \text{rank}(\varphi) + 1$.

(iii) If $S = \{\varphi_n\}$ is a countable set of formulae, then $\bigwedge_{n\in\omega} \varphi_n$ is also a formula: This is interpreted simply as the conjunction of all the claims in $S$. If $\psi = \bigwedge_{n\in\omega} \varphi_n$ then $\text{rank}(\psi) = \sup\{\text{rank}(\varphi_n) + 1 : n \in \omega\}$.

Then for a tree $T$ and a formula $\varphi$ as defined above, we write $\varphi(T)$ if $T$ satisfies the claims made by $\varphi$.

Now that we've built the language and rank function of our language, we can build the forcing relation.

**Definition 4.16.** Given $p \in Q_\alpha$, and $\varphi$ a formula in the language of Steel forcing, we define $p \Vdash_\alpha \varphi$ by:

(i) $p \Vdash_\alpha \eta \in T'$ iff $p = (t, h)$ and $\eta \in t$, or $p = (t, h)$ and $\exists \tau \in t$ such that $h(\tau) \neq 0$, and $\eta$ is an extension of $\tau$ by one element (i.e. $\tau \subseteq \eta$ and $|\eta \setminus \tau| = 1$).

(ii) $p \Vdash_\alpha \neg\varphi$ iff $q \nVdash \varphi$ for all $q \leqslant_\alpha p$.

(iii) $p \Vdash_\alpha \bigwedge_{n\in\omega} \varphi_n$ iff $\forall n \in \omega \;\; p \Vdash_\alpha \varphi_n$.

We can identify any filter $G$ on $Q_\alpha$ with an associated tree, $T$, and height function, $H_T$, on ${}^{<\omega}\omega$ via setting $T = \bigcup\{t : (t, h) \in G\}$, and similarly $H_T = \bigcup\{h : (t, h) \in G\}$.

**Definition 4.17** (Sufficiently $Q_\alpha$-generic)**.** We say that a tree is **sufficiently** $Q_\alpha$**-generic** iff there is some countable sequence of dense subsets $(D_n)_{n\in\omega}$ of

$Q_\alpha$ such that a filter $G$ meets all $D_n$, and $T$ extends every condition in that filter.

To clarify the role of the $(D_n)_{n\in\omega}$ in this definition, if there is a property which we want a tree to have and we can express this property as meeting some countable collection of dense subsets of $Q_\alpha$, then we can say that *any sufficiently* $Q_\alpha$*-generic* tree will have that property.

**Definition 4.18** (Extensions of conditions)**.** We say that a tree $T$ **extends a condition** $(t, h) \in Q_\alpha$ if we have $t \subseteq T$, and whenever $\eta \in T$ with $h(\eta) \neq \infty$, and $H_T(\eta) \leqslant \omega \cdot \alpha$, then $H_T(\eta) = h(\eta)$

With these definitions in hand, we can return to and explain a confusing part of definition 4.16. Why not define the forcing for atomic formulae such that $(t, h) = p \Vdash_\alpha \eta \in T'$ iff $\eta \in t$. The reason for the additional "extend-by-one" condition is that if $h(\tau) \neq 0$, and $\eta$ is an extension-by-one of $\tau$, then any sufficiently generic tree extending $p = (t, h) \in Q_\alpha$ must also include $\eta$. So if $\eta$ can be obtained via extending-by-one, then it will hold for any $Q_\alpha$-generic tree $T$ extending $p$, and thus be forced by $p$. This follows from the following proposition.

**Proposition 4.19.** *For any* $p = (t, h) \in Q_\alpha$*,* $p \Vdash_\alpha \varphi$ *iff* $\varphi(T)$ *is true for all sufficiently* $Q_\alpha$*-generic trees* $T$ *extending* $p$*.*

This proposition is proved in [11, Thm 3.5].

**Definition 4.20** (Generic over a transitive set)**.** Let $M$ be a transitive set, we say that a tree $T$ is **generic over** $M$ if for all sentences $\varphi \in M$ such that $\text{rank}(\varphi) = \alpha$, there is a $p \in Q_\alpha$ such that $T$ extends $p$, and $p \Vdash_\alpha \varphi$. We say that a sentence $\varphi$ is "in" a transitive set $M$ by fixing a bijection between $\omega$ and the (countable) set of symbols of our forcing language, then $\varphi$ is in $M$ iff the preimage of $\varphi$ (which will consist of a countable sequence of elements) under this bijection is in $M$.

**Lemma 4.21.** *For any countable transitive set, and forcing poset* $Q_\alpha$*, there is a tree* $T$ *which is generic over* $M$*.*

*Proof.* To show that $T$ is generic over a countable transitive set $M$, we just need to show that for all $\alpha \in \text{Ord} \cap M$, $T$ extends every condition in a filter on $Q_\alpha$ which intersects every dense subset of $Q_\alpha$ which is a set in $M$. This will mean there is a $p \in Q_\alpha$ such that $p \Vdash_\alpha \varphi$ (since every $\varphi \in M$ must have $\text{rank}(\varphi)$ equal to some ordinal in $M$). Since we have taken $M$ to be a countable transitive set, there are only countably many dense subsets of $Q_\alpha$ which lie in $M$. Then also there are only countably many ordinals to consider, so we can inductively build a tree which extends each condition in $\omega$ steps. $\square$

### 4.2.2 An analytic set of reals

In order to use analytic determinacy to prove HP holds (and therefore that $0^{\#}$ exists) we will utilise a carefully chosen analytic set. Then, applying the determinacy condition will yield the result. In this section, we will define such a set and show that it is analytic, and then move on to establish results which will aid in showing that the determinacy of this set implies HP. We utilise results from [6] and [5].

**Definition 4.22** (End Extension)**.** If $(A, R)$ and $(B, S)$ are sets with binary relations $R$ and $S$, and $A \subseteq B$, then we say that the relation $S$ is an **end extension** of $R$ if $S \restriction A = R$ and for all $b \in B \setminus A$, there is no $a \in A$ such that $(bSa)$.

Recall definition 4.7, using this notion, we introduce:

**Definition 4.23** (The set of reals $A$)**.** Define $A \subseteq {}^{\omega}\omega$ by: $a \in A$ iff there exists $(\omega, R)$ such that $R$ is recursive in $a$, and $(\omega, R)$ is isomorphic to an end extension of $(L_{\omega_1(a)}, \in)$.

**Proposition 4.24.** *$A$ is $\Sigma^1_1$.*

*Proof.* The set of $(a, R)$ such that "$R \subseteq \omega \times \omega$ is recursive in $a$" is $\Delta^1_1$ (i.e. lies in the arithmetical hierarchy). To demonstrate this, first observe that $\{R : R$ is recursive in $a\}$ is naturally a $\Sigma^0_3(a)$ set, since it claims that

"There is a Turing machine $M$ with oracle $a$ so that for every pair $(n, m)$, there is a time $t$ by which $M$ has computed whether $(n, m) \in R$."

This description is $\Sigma^0_3(a)$, since in order to compute whether a relation satisfies the above criteria, we will need an oracle for $a$. However, when we consider the set $\{(a, R) : R$ is recursive in $a\}$, we will not need an oracle for $a$. Then this set is then $\Sigma^0_3$, since in order determine whether a *given* pair $(a, R)$ is such that $R$ is recursive in $a$, we will not need an oracle for $a$. Therefore this set is $\Delta^1_1$.

Then, we have that $(\omega, R)$ is isomorphic to an end-extension of $L_{\omega_1(a)}$ iff two conditions hold. Firstly, $(\omega, R)$ itself has an end extension $(B, S)$ with an element $b \in B$ such that

$$a = \{n \in \omega : (n^B, b) \in S\}$$

where $n^B$ is defined in $(B, S)$ in the same way that $n$ is defined as a set in $(V, \in)$. For example, 0 is defined as the element such that there is no $y$ with $(y, 0) \in S$, then 1 is defined as the element such that $(0, 1) \in S$ and there is no other $y$ such that $(y, 1) \in S$, and so on. This means $b$ is a copy of $a$ inside $B$.

The second condition necessary for $(\omega, R)$ to be isomorphic to an end-extension of $L_{\omega_1(a)}$ is that there is an ordinal $\alpha \in B$ such that $(L_\alpha[b])^B$ is an admissible set, and $L^B_\alpha$ exists inside of $(\omega, R)$.

These conditions suffice to show that $(\omega, R)$ is isomorphic to an end-extension of $L_{\omega_1(a)}$. We need only consider extensions $(B, S)$ which are countable, as if there is some extension, there will be some countable extension. So in the definition of $A$, when we claim that there exists such a $(B, S)$, this will correspond to an existential quantifier which ranges over the reals (which will be taken to code such an extension), followed by the $\Delta^1_1$ description of $(a, R)$. Therefore, we can express the statement that "$(\omega, R)$ is isomorphic to an end extension of $L_{\omega_1(a)}$ and $R$ is recursive in $a$" as a $\Sigma^1_1$ sentence, and so therefore the set of these form a $\Sigma^1_1$-set. Therefore $A$ is (lightface) analytic. $\square$

We can also see that the set $A$ is closed under Turing equivalence, since if $a \in A$ and $a \equiv_{\mathrm{T}} b$, this implies $\omega_1(a) = \omega_1(b)$, since all ordinals computable relative to $a$ are computable relative to $b$, and vice versa. Then if a relation $R$ on $\omega$ is recursive in $a$ and extends $L_{\omega_1(a)}$ it is also recursive in $b$. This will be important in section 4.2.3, where we will use a result of Martin's about determined sets which are closed under Turing equivalence.

We now proceed to show that $A$ is non-empty. We prove this by demonstrating that for any $a \in {}^\omega\omega$ (i.e. any real number) there is some $b \in {}^\omega\omega$ such that $\langle a, b\rangle$, the *Turing join* of $a$ and $b$, is in $A$. By the Turing join, we mean a real $r \in {}^\omega\omega$ with $r(2i) = a(i)$ and $r(2i+1) = b(i)$. For this result, we will need two results which we won't endeavour to prove, but we will indicate where the relevant proofs can be found.

**Theorem 4.25.** *[9, Ch. 20, Thm. 26] Let $U$ be a countable model of $\mathsf{ZF} + V = L$. Then $U$ has an elementary end-extension, $U'$ of arbitrarily large cardinality with a countable sequence of indiscernibles $(d_i)_{i\in\omega}$ such that $U' \vDash d_i$ is an ordinal for each $i \in \omega$, and for $i < j$, $U' \vDash d_i \in d_j$.*

In fact, the proof Keisler gives does not need the full strength of a model of $\mathsf{ZF} + V = L$. We need only that $U$ is a model of a sufficiently large fragment of $\mathsf{ZF} + V = L$ so that the proof goes through. Additionally, the same argument that works for $V = L$ can be transferred without difficulty to $V = L[a]$.

**Theorem 4.26.** *[9, Ch. 13, Thm. 19(iii)] Suppose $U$ models a countable fragment $\mathcal{L}_A$ of $\mathcal{L}_{\omega_1,\omega}$ - the language with countably infinite conjunctions and disjunctions, but only finite quantification[10]. If there is an infinite set of order-indiscernibles $(X, <)$ for $U$, then for any infinite linearly ordered set $(Y, <)$, there is a model $B$ with order-indiscernibles $(Y, <)$ such that any finite increasing sequence from $X$ and $Y$ of the same length realise the same types.*

**Lemma 4.27.** *For all $a \in {}^\omega\omega$, there is some $b$ such that $\langle a, b\rangle \in A$.*

*Proof.* Let $\beta$ be a countable ordinal, such that $L_\beta[a]$ models a sufficient fragment of $\mathsf{ZF} + V = L[a]$. Let this fragment be such that $\beta$ is $a$-admissible. Then by

[10] And in our context, equality and set-inclusion symbols.

theorem 4.25 above, there is a countable elementary end extension $U'$ of $L_\beta[a]$ so that $U'$ has a sequence of order indiscernibles. Then applying theorem 4.26 to $U'$, and taking a set of indiscernible $(Y, <)$ with order-type the rationals, we get a (possibly different) elementary end extension of $L_\beta[a]$, $(B, E)$, with a set $Y$ of indiscernibles with order-type of the rationals. This can be done by using a fragment $\mathcal{L}_A$ of $\mathcal{L}_{\omega_1,\omega}$ which is sufficiently expressive to ensure that every model of $\mathcal{L}_A$ is (isomorphic to) an elementary end extension of $L_\beta[a]$. This can be done since we are allowed to take countably infinite conjunctions and $\beta$ is a countable ordinal, so $L_\beta[a]$ is also countable.

Then $B$ must be ill-founded, since it contains a set of ordinals of order-type the rationals. However we can take the *well-founded-core* of $(B, E)$, by which we mean the set of all $x \in B$ such that $E \cap \mathrm{cl}(x) \times \mathrm{cl}(x)$[11] is well-founded. Then this well-founded core is well-founded and satisfies extensionality, so it is isomorphic to a transitive structure. Then $B$ also is isomorphic to a structure with a transitive well-founded core.

Then $B$ is an $\omega$-model, i.e. all the natural numbers in $B$ have the same definition as the natural numbers in $\omega^V$.However, the ordinals of $B$ must be different than the ordinals in $L_\beta[a]$ since $B$ is ill-founded, so we can take a non-standard $\alpha \in B$ such that $B$ believes $\alpha$ is countable. Then also in $B$, there is some relation $b \subseteq \omega \times \omega$ which is isomorphic to $L^B_\alpha$, since $B \vDash L^B_\alpha$ is countable, and therefore that $L^B_\alpha$ is isomorphic to the structure of a relation on $\omega$. Then $a$ and $b$ exist in $B$, and in fact lie in the well-founded core of $B$, since $B$ is an $\omega$-model. Since $B$ is admissible, (due to being an *elementary* end-extension of $L_\beta[a]$ for some $\beta$ which is $a$-admissible), the well-founded core of $B$ must also be admissible [13, Prop 6.39]. Therefore, we must have $\omega_1(a, b)$, the *least* ordinal which is $a$- and $b$-admissible, is a subset of $B$, and in fact a subset of the well-founded core of $B$. Therefore, for all $\xi < \omega_1(a, b)$, $L^B_\alpha$ is an end-extenson of $L_\xi$, since $\alpha > \xi$ in $B$. So $L_\alpha$ end-extends $L_{\omega_1(a,b)}$, and therefore we have $\langle a, b\rangle \in A$. $\square$

**Proposition 4.28.** *Take $a \subseteq \omega$, and $\rho$ a countable ordinal. Assume that for every countable ordinal $\xi$, there is a tree $T$ which is generic over $L_\xi[a]$, and that there is some $b \subseteq \omega$ so that $b \in L_\rho[T, a]$ and $b \in A$. Then* HP *holds.*

However, before we can prove this result, we need a lemma.

**Lemma 4.29.** *Let $b \in A$, and $\gamma < \xi < \omega_1(b)$. Then also let $X \subseteq \gamma$ so that $X \in L_\xi$. Then $X \in L_{\gamma \cdot 3}$*

*Proof.* Since $\xi < \omega_1(b)$, $\xi$ is computable relative to $b$, so there is a well-founded relation $R \subseteq \omega \times \omega$ which has order-type $\xi$. We can construct $(\omega, R)$ in $L_{\omega+7}[b]$. Then also $b \in A$, so there is a relation $(\omega, S)$ which end-extends $L_{\omega_1(b)}$, and therefore this relation $S$ end-extends $(L_\xi, \in)$. The relation $S$ can be defined in

[11]We write $\mathrm{cl}(x)$ to denote the closure of $\{x\}$ under $E$, defined recursively as $\mathrm{cl}(x) = \{x\} \cup \bigcup\{\mathrm{cl}(y) : yEx\}$.

$L_{\omega+8}[b]$ by recursing along $R$. Then inside $(\omega, S)$ there is an equivalent of $\gamma$ and of $X$ whch we will call $\gamma'$ and $X'$. Then, since $S$ is recursive in $b$, $S$ lies in $L_{\gamma+1}[b]$. Moreover, we can construct the isomorphism between $\gamma'$ and $\gamma$ inside $L_{\gamma\cdot 3}$ and this isomorphism maps $X'$ onto $X$, so $X \in L_{\gamma\cdot 3}$. □

*Proof of proposition 4.28.* Let $\alpha > \rho$ be a countable $a$-admissible ordinal. Then it suffices to show that $\alpha$ is an $L$-cardinal, since this will imply HP (as $\rho$ was arbitrary). To demonstrate this, it is enough to show that for all $\gamma < \alpha$, $(\gamma^+)^L \leqslant \alpha$[12]. Every ordinal below $(\gamma^+)^L$ is encoded by a (constructible) ordering of $\gamma$. If we can code an ordinal inside of an admissible set, then KP is strong enough to prove that the ordinal itself must exist, and so it must already lie in the admissible set. Therefore it is enough to show that any constructible $X \subset \gamma$ has $X \in L_\alpha[a]$.

Take a countable ordinal $\beta$ such that $X \in L_\beta$, and then take a countable $\xi$ with $\xi > \beta, \alpha$. Then since $\xi$ is countable, the conditions of the proposition guarantee that there is a tree $T$ generic over $L_\xi[a]$ and a $b \subseteq \omega$ such that $b \in L_\rho[a, T]$, $b \in A$, and $\omega_1(b) > \beta$.

Then, using lemma 4.29 and the conditions on $b$, we have that $X \in L_{\gamma\cdot 3}$. Then since $b \in L_\rho[a, T]$ and $\alpha > \gamma \cdot 3, \rho$[13], $L_\alpha[a, T]$ contains $b$ and therefore also contains $X$ (since $X$ is definable relative to $b$, and thus also in $L_\alpha[a, T]$ if $b \in L_\alpha[a, T]$). Therefore, we know that in $L_\alpha[a]$, we can define $X$ from an initial segment of $T$ of height $< \alpha$.

Therefore, there is some $\delta < \alpha$ such that, from the forcing poset $Q_\delta$, we can construct a term $\sigma \in L_\alpha[a]$ so that $X = \sigma(T)$. Then in the forcing language defined in section 4.2.1, we can express $X = \sigma(T)$ as a sentence $\varphi$ by:

$$\varphi := \left( \bigwedge_{i \in X} i \in \sigma(T') \right) \wedge \left( \bigwedge_{i \in \kappa \setminus X} i \notin \sigma(T') \right)$$

Then each formula"$i \in \sigma(T')$" is of rank $< \alpha$ since $\sigma$ is a term in $L_\alpha[a]$. Let $\eta := \text{rank}(\varphi)$, with $\eta < \alpha$. Then recall that the tree $T$ given to us by the hypothesis of the proposition is generic over $L_\xi[a]$, so we have some $p \in Q_\eta$ such that $p \Vdash_\eta \varphi(T)$. Then we can express $X$ as

$$X = \{i \in \kappa : \exists q \leqslant_\eta p \; q \Vdash_\gamma (i \in \sigma(T'))\}.$$

Then finally, since $p \in Q_\eta$ and $\eta < \alpha$, the $p$ which forces this lies in $L_\alpha[a]$, and so we have that $X \in L_\alpha[a]$. □

### 4.2.3 Analytic Determinacy implies $0^\#$

With proposition 4.28, we are almost at our result. The last step is to find an $a \subseteq \omega$ that satisfies the conditions of the proposition. We will use our condition

[12]By $(\gamma^+)^L$, we mean the smallest $L$-cardinal greater than $\gamma$.

[13]We have that $\alpha$ is greater than $\gamma \cdot 3$, since $\alpha$ is admissible and $\gamma < \alpha$. If $M$ models KP and $\delta \in M \cap \text{Ord}$, then also $\delta \cdot 3 \in M \cap \text{Ord}$

of analytic determinacy to find such an $a$, but first we need to introduce a basic definition, and one result tying together infinite games and computability.

**Definition 4.30** (Turing cone)**.** We say that a set $A \subseteq \mathcal{P}(\omega)$ of subsets of the natural numbers is a **Turing cone** if there is some $a \subseteq \omega$ such that for all $b \subseteq \omega$, $b \in A$ iff $a \leqslant_{\mathrm{T}} b$. We call $a$ the *base* of the Turing cone.

Then also we can define Turing cones on subsets of ${}^{\omega}\omega$ via an identification between ${}^{\omega}\omega$ and $\mathcal{P}(\omega)$. Now we can prove Martin's theorem.

**Theorem 4.31** (Martin's Theorem)**.** *If a game with payoff set $A \subseteq {}^{\omega}\omega$ is determined, and $A$ is closed under Turing equivalence, then $A$ or ${}^{\omega}\omega \setminus A$ contains a Turing cone.*

*Proof.* Suppose player I has a winning strategy given by $f : {}^{<\omega}\omega \to \omega$. This can be coded by an element of ${}^{\omega}\omega$, and moreover, we can construct such a coding which is computable[14]. Then let $a \in {}^{\omega}\omega$ be the real coding $f$. Since this coding is computable, $a \equiv_{\mathrm{T}} f$. Then for $b \geqslant_{\mathrm{T}} a$ and $b \in {}^{\omega}\omega$, let $c$ be the run of the game where player I plays $f$ and player II plays the strategy coded by $b$, so that we can write $c = f * b$. But then

$$b \leqslant_{\mathrm{T}} f * b \equiv_{\mathrm{T}} a * b \leqslant_{\mathrm{T}} b$$

Therefore, since $f * b \in A$ as $f$ was a winning strategy for player I, we must have $b \in A$ by closure under Turing equivalence. The argument for player II is virtually identical □

*Proof of theorem 4.1.* Let $A$ be the set from section 5.2.2. We know that for any $a \subseteq \omega$, there is some $b \subseteq \omega$ so that $a * b \in A$. Therefore if $x \in {}^{\omega}\omega \setminus A$, there is some $y$ such that $x * y \in A$, so ${}^{\omega}\omega \setminus A$ cannot contain a Turing cone and therefore, by analytic determinacy and Martin's theorem, $A$ must contain such a cone.

Let $a \subseteq \omega$ be the base of this cone. Then let $\rho$ be a countable ordinal, and for a countable $\xi$, take a tree $T$ which is generic over $L_{\xi}[a]$, which must exist by lemma 4.21. Then if we set $b = a * r(T)$ (where $r(T)$ is the real corresponding to $T$), then $b \in A$ since $b$ is in the cone with base $a$. Further, from the definition of $b$, we can construct $b$ from $a$ and $T$, so for reasonably large $\rho$, we have $b \in L_{\rho}[a, T]$. Finally, $\omega_1(b) \geqslant \xi$, since $T$ is recursive relative to $b$, and since $T$ is generic over $L_{\xi}[a]$, we can use $T$ (and therefore $b$) to compute $\xi$. Therefore $a \subseteq \omega$ satisfies the conditions of propositions 4.28, so HP holds. So $0^{\#}$ exists. □

[14] A computable bijection ${}^{<\omega}\omega \to \omega$ will suffice, as then we can transfer this to a bijection between functions from ${}^{<\omega}\omega \to \omega$, and functions $\omega \to \omega$. This bijection can be built, for example, using prime factors of elements of $\omega$ to encode elements of ${}^{<\omega}\omega$, and can be done in a computable way

### 4.3 Relativizing Harrington's Theorem

The above proof shows that from $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$ we can prove that $0^\#$ exists. However, in our proof of Harrington's principle, we only required $\mathrm{Det}(\Sigma^1_1)$: we constructed $A$ which is a $\Sigma^1_1$ set, and used $\Sigma^1_1$-determinacy to show that $\mathsf{HP}$ holds. Then from $\mathsf{HP}$, we demonstrated in section 4.1 that $0^\#$ exists.

Of course, $\mathrm{Det}(\Sigma^1_1)$ follows from $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$ by the remark following proposition 2.11. However, so does $\mathrm{Det}(\Sigma^1_1(x))$ for all $x \subseteq \omega$. So we can actually relativise the entire proof we just gave. We'll briefly indicate the changes that would need to be made to relativise the proof.

Harrington's principle would relativise to $\mathsf{HP}(x)$, which states that there exists $a \subseteq \omega$ such that an $a$-admissible ordinal is an $L(x)$-cardinal. Then from $\mathsf{HP}(x)$, the proof of lemma 4.10 would proceed identically, instead constructing an ultrafilter on $L_{\omega_2}(x)$. Then combining corollary 3.17 and the remark following definition 3.15 gives us that $x^\#$ exists.

The proof of $\mathsf{HP}(x)$ from $\mathrm{Det}(\Sigma^1_1(x))$ will then utilise a $\Sigma^1_1(x)$ set, instead of our $\Sigma^1_1$ set $A$. This set contains $a \subseteq \omega$ iff there is a relation on $\omega$ recursive in $a$ which end-extends $L_{\omega_1(a)}(x)$. Then this set is $\Sigma^1_1(x)$, by almost the same argument that shows $A$ is $\Sigma^1_1$, with a minor addition demonstrating that we will need an oracle for $x$ in order to determine whether the proposed extension is actually an end-extension.

Proposition 4.28 remains mostly unchanged, except for requiring that $b$ is in our $\Sigma^1_1(x)$ set instead of our original $A$. Finally, we use $\Sigma^1_1(x)$-determinacy and satisfy the conditions of proposition 4.28, just as before, and we get $\mathsf{HP}(x)$ and therefore $x^\#$ exists.

## 5 Consistency Strength

Finally, we shall discuss the consistency strengths of certain large cardinal hypotheses, and how they compare to the existence of $0^\#$ and analytic determinacy. In order to do this, we will state (but not prove) the Martin-Harrington theorem, which encompasses both the result we have just proven, as well as the reverse implication. First, let's establish some additional large cardinal hypotheses which will be useful in our discussion

**Definition 5.1** (Measurable Cardinal)**.** We say that a cardinal $\kappa$ is a **measurable cardinal** if there is a $\kappa$-complete, non-principal ultrafilter on $\kappa$. (For more information about measurable cardinals, see [8, after 2.7].)

**Definition 5.2** (Ramsey Cardinal)**.** We say that a cardinal $\kappa$ is a **Ramsey cardinal** if:

$$\kappa \to (\kappa)^{<\omega}_2,$$

that is, for any colouring of finite subsets of $\kappa$ by 2 colours, there is a homogeneous subset $H \subseteq \kappa$ such that $|H| = \kappa$. (For more information about Ramsey cardinals, see [8, after 7.12].)

**Theorem 5.3.** $\mathsf{ZFC} + \mathsf{MC}$ *implies* $\mathrm{Cons}(\mathsf{ZFC} + 0^{\#}$ *exists*$)$.

*Proof.* It is well-known (see e.g. [8, Prop 5.7]) that the existence of a measurable cardinal $\kappa$ implies that for some transitive class $M$, there is an elementary embedding $j : V \to M$ with critical point $\kappa$. Then let $x$ be any set, and suppose that $x^V = x^M$. Then $L(x)^V = L(x)^M$. This is essentially follows from the proof that being constructible is a $\Delta_1$ property (for a proof of this, see [7, Lem. 13.14]), and therefore absolute for an inner model of $\mathsf{ZF}$, just changing the starting point of this from $\emptyset$ to the transitive closure of $x$.

Then since $j : V \to M$ is *elementary*, when we restrict the map to $L(x)$ in $V$, we obtain a map to $L(x)$ in $M$, since elements of these classes are all first-order definable, and thus preserved by elementary embeddings. However, $\kappa \in L(x)$, and $\kappa = \mathrm{crit}(j)$, so we have a nontrivial elementary embedding, $e : L(x) \to L(x)$, and therefore that the sharp of any $x \subseteq \omega$ exists. In particular, since $0^{\#} \subseteq \omega$, we have that $0^{\#\#}$ exists and so by the definition of $0^{\#}$ given in definition 3.15, we have a definable truth predicate for $L[0^{\#}]$, and therefore a model of $\mathsf{ZFC} + 0^{\#}$ exists. Therefore, $\mathsf{ZFC} + \mathsf{MC}$ implies $\mathrm{Cons}(\mathsf{ZFC} + 0^{\#}$ exists$)$. □

Of course, this immediately gives that $\mathsf{ZFC} + \mathsf{MC}$ is has strictly greater consistency strength than $\mathsf{ZFC} + 0^{\#}$ exists.

**Remark.** We can get a sharper result than going from a measurable cardinal. Jech [7, after Cor. 18.29] gives a proof of the existence of $0^{\#}$ exists from a Ramsey cardinal. And in fact, if we have a Ramsey cardinal $\kappa$, then all $a \subseteq \omega$ has $a^{\#}$ exists, essentially the proof Jech gives works for $L[a]$ for all $a \in \omega$ (although he only gives it for $0^{\#}$).

So far we've only discussed Harrington's proof of "$0^{\#}$ exists" from $\mathrm{Det}(\Sigma^1_1)$. However, there is a remarkable counterpart to this result, which is that we can *also* prove $\mathrm{Det}(\Sigma^1_1)$ from $0^{\#}$ existing! The result we've proved here is therefore actually an equivalence. These two results are known collectively as the Martin-Harrington theorem. Yong Cheng [2, after Thm 1.23] separates them into two results as follows.

**Theorem 5.4** (Martin-Harrington theorem)**.**

*(i) The* lightface *Martin-Harrington theorem states that* $\mathrm{Det}(\Sigma^1_1)$ *holds iff* $0^{\#}$ *exists.*

*(ii) The* ***boldface*** *Martin-Harrington theorem states that* $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$ *holds iff for all* $x \subseteq \omega$*,* $x^{\#}$ *exists.*

Proving Martin's theorem is outside of the scope of this essay, but for a good discussion and proof of Martin's (boldface) theorem see [2, §3.1]. However, using Martin's theorem together with theorem 5.3, we can observe the following corollary.

**Corollary 5.5.** *Lightface analytic determinacy does not prove the existence of a measurable cardinal.*

*Proof.* Let $M$ be a countable transitive model of $\mathsf{ZFC} + \mathsf{MC}$. Let $\kappa \in M$ be a measurable cardinal, and without loss of generality let $M$ model that there is only one measurable cardinal. Let $\mathrm{Col}(\omega, \kappa)$ be the collapsing poset consisting of all partial functions from $\omega$ to $\kappa$ of cardinality less than $\omega$, and let $G \subseteq \mathrm{Col}(\omega, \kappa)$ be generic over $M$. Then $M[G] \vDash \mathsf{ZFC} + \neg\mathsf{MC}$ since the only measurable cardinal in $M$ has been collapsed. We have that $M \vDash \mathsf{ZFC} + \mathsf{MC}$, and therefore by 5.3 $M \vDash$ "$0^\#$ exists".

We also have that $0^\#$ is preserved under forcings, since it is a subset of $\omega$ with a definition which is absolute. Therefore $M[G] \vDash 0^\#$ exists and so, by theorem 5.4, $M[G] \vDash \mathrm{Det}(\Sigma^1_1)$. However, $M[G]$ does not model $\mathsf{MC}$, therefore it cannot be the case that lightface analytic determinacy implies the existence of a measurable cardinal. □

**Remark.** The tactic of the above proof also shows that "$0^\#$ exists" (equivalently $\mathrm{Det}(\boldsymbol{\Sigma}^1_1)$) cannot be equivalent to any proper large cardinal axiom[15]. If it were then we could proceed with the same proof, collapsing this cardinal in a generic extension, but preserving $0^\#$.

**Remark.** We also have that boldface analytic determinacy is not sufficient to prove the existence of a measurable cardinal. The remark after theorem 5.3 tells us that all we really need is a Ramsey cardinal to prove the required consistency hypothesis, and the least Ramsey cardinal is not measurable [8, p. 7.19], so if $\kappa$ is measurable, then $V_\kappa \vDash \exists\lambda$, $\lambda$ is Ramsey. From this, we get that $\mathsf{ZFC} + \mathsf{MC} \vdash \mathrm{Cons}(\mathsf{ZFC} + \forall a \subseteq \omega\ a^\#$ exists), and so by the boldface version of Martin's theorem, also $\mathrm{Cons}(\mathsf{ZFC} + \mathrm{Det}(\boldsymbol{\Sigma}^1_1))$.

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